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part 2

ON FINITE AND INFINITE C^k-$\mathcal{A}$ DETERMINACY

by

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§4. The set of good points

In this section we will state some technical results, and show that these results imply parts of Theorem 0.1. Let us start with the following proposition.

Proposition 4.1. Let $f:(R^n,0) \to (R^p,0)$ be a C^∞ map-germ. Let either k be a non-negative integer or $k = \infty$. Suppose that for each such k there exists $m = m(k)$, which is a non-negative integer if $k<\infty$ or $m(k) = \infty$ if $k = \infty$, such that the following holds:

Let $h:(R^n,0) \to (R^p,0)$ be a C^∞ map-germ such that $j^m h(0) = 0$. Let $I = [0,1]$, and consider the map-germ $F:(R^n\times R, \{0\}\times I) \to (R^p\times R, \{0\}\times I)$ defined by $F(x,t) = (f(x)+th(x),t)$. Then there exist germs at $\{0\}\times I$ of C^∞ vectorfields X,Y defined on $R^n\times R-\{0\}\times I$, $R^p\times R-\{0\}\times I$ respectively such that X,Y are of the following form;

$$X(x,t) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x,t)\frac{\partial}{\partial x_i}, \quad Y(y,t) = \frac{\partial}{\partial t} + \sum_{j=1}^{p} Y_j(y,t)\frac{\partial}{\partial y_j},$$

and such that X,Y satisfy the following two conditions below:

(1) $DF(X) = Y\circ F$

(2) Consider multiindices $(\alpha,s) = (\alpha_1,\ldots,\alpha_n,s)$, $(\beta,q) = (\beta_1,\ldots,\beta_p,q)$ such that $|\alpha|+s\leqslant k$, $|\beta|+q\leqslant k$. (If $k = \infty$, we impose no restrictions on (α,s), (β,q).) Then if $k<\infty$ and (α,s), (β,q) are two such multiindices, there exist neighbourhoods of $\{0\}\times I$ such that for each $1\leqslant i\leqslant n$, $1\leqslant j\leqslant p$ we have:

$$\frac{\partial^{|\alpha,s|}}{\partial^{(\alpha,s)}(x,t)} X_i(x,t) = \frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^s}{\partial t^s} X_i(x,t) \leqslant \|x\|^{k+1}$$

$$\frac{\partial^{|\beta,q|}}{\partial^{(\beta,q)}(y,t)} Y_j(y,t) = \frac{\partial^{|\beta|}}{\partial y^\beta}\frac{\partial^q}{\partial t^q} Y_j(y,t) \leqslant \|y\|^{k+1}.$$

If $k = \infty$, then for each positive integer ℓ, and each pair of multiindices, there exist neighbourhoods of $\{0\}\times I$, such that the inequalities we get by replacing k with ℓ in (2) hold in these neighbourhoods.

Then, if f satisfies the above hypothesis, f is $m(k)$-$\mathcal{A}^{(k)}$ determined.

Proof. Pick representatives of f, h, X and Y satisfying the hypothesis in the Proposition. Extend X and Y to $R^n\times R$, $R^p\times R$ by putting $X(0,t) = \frac{\partial}{\partial t}$, $Y(0,t) = \frac{\partial}{\partial t}$. Then it follows easily from (2) that X and Y are C^k at points in $\{0\}\times I$. Picking representatives of X and Y defined on some neighbourhoods of $\{0\}\times I$, we may integrate X and Y locally. Since the t components of X and Y are 1, and the X_i, Y_j components are small because of (2), all the flowlines starting at points in some small neighbourhood of $\{0\}\times I$ will reach the levels $R^n\times\{1\}$, $R^p\times\{1\}$. Integrating X and Y, we thus get a 1-parameter family of C^k-diffemorphisms $\{(\phi_t,\psi_t)\}$ of R^n, R^p where t varies in an interval containing $[0,1]$. Because of (1) the flow lines of X are mapped by F to the flowlines of Y. This implies that $f_t\circ\phi_t = \psi_t\circ f_0$ (where $f_t(x) = f(x)+th(x)$). From this we get that $f_1 = f+h$, and $f_0 = f$ are $\mathcal{A}^{(k)}$-equivalent. Since h is an arbitrary germ with $j^m h(0) = 0$, f is m-$^{(k)}$determined.

The case $k = 0$, will need some extra arguments to show that (2) implies that a C^0 flow exists. However, since we never will bother to estimate degrees of determinacy, and $(b_k) \Rightarrow (b_0)$ for $k>0$, we omit the details needed in this special case.

In the rest of the article we will show that if f satisfies $L_1(f),\ldots,L_{p+1}(f)$, then for each k there exists an $m = m(k)$

such that the hypothesis of 4.1 holds. We thus have to show that for each k, there exists an $m(k)$, such that if $j^{m(k)}h(0) = 0$ then vectorfields satisfying (1) and (2) of 4.1 exist. The construction of X,Y will be done locally on $R^n \times R - \{0\} \times I$, $R^p \times R - \{0\} \times I$, and we will then glue the locally defined vectorfields together with a suitable partition of unity. To construct X and Y locally such that (1) is satisfied, we will use Proposition 2.5. We will thus construct X and Y by the one sided inverse σ to the map $tf_t + \omega f_t$. To be able to do so, we need to know that each level f_t is stable outside 0. For each k we therefore choose $m(k) > m$, where m is the m of Proposition 3.1. It will follow from this proposition that each level f_t satisfies $L_1(f_t), \ldots, L_{p+1}(f_t)$, which implies that f_t is stable outside 0. To show that X,Y can be chosen such that (2) is satisfied, we will use the estimates of ξ_t and η_t in Proposition 2.5. Since powers of a and b occur in the denominators of these estimates, it turns out that we need local solutions of X and Y at points where the corresponding a,b are not too small. Actually a and b have to satisfy Lojasiewicz inequalities with respect to the distance to $\{0\} \times I$. It thus follows that we have to construct our local solution of X and Y at a rather special set of "good" points in $R^n \times R - \{0\} \times I$, $R^p \times R - \{0\} \times I$. Since we want to use a partition of unity argument to obtain global solutions, we need to show there exists enough "good" points such that the neighbourhoods (centered at these points) where the local solution are constructed also cover $R^n \times R - \{0\} \times I$, $R^p \times R - \{0\} \times I$. If too many supports of partition functions intersect, or the derivatives of the partition functions are too large, (2) of 4.1 can be destroyed when we glue together the local solutions. Our "good" points therefore need to have the property that this will not happen.

The above is ment as motivation for the definition of "good" points we now are going to give. The next proposition will give the definition of the "good" set of points, and will claim the existence of this set.

Proposition 4.2. Suppose $f:(R^n,0) \to (R^p,0)$ satisfies $L_1(f),\ldots,L_{p+1}(f)$. Let m be the integer of Proposition 3.1. Let $h:(R^n,0) \to (R^p,0)$ be such that $j^m h(0) = 0$. Consider the map-germ at $\{0\}\times[0,1] \subset R^n\times R$ defined by $F(x,t) = (f_t(x),t)$ where $f_t(x) = f(x)+th(x)$. In the above situation the following condition are satisfied:

There exist representatives of f and h defined on a common neighbourhood U, such that the induced representative of each f_t satisfies $L_1(f_t),\ldots,L_{p+1}(f_t)$. There will also exist a set $\Lambda \subset F(\sum(F))-\{0\}\times[0,1]$ of "good points" satisfying the following:

For each $(y^0,t^0)\in\Lambda$ there exist positive constants $C,C',\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5$ such that the following set of conditions hold.

Let $S = f_{t^0}^{-1}(y^0)\cap\sum(f_{t^0}) = \{x_1^0,\ldots,x_\ell^0\}$, and let $x^0 = (x_1^0,\ldots,x_\ell^0)\in(R^n)^\ell$. Then we have:

(1) $d(x^0,D_\ell(f_{t^0})) \geq \min\limits_{1\leq i\leq \ell} \|x_i^0\|^{\lambda_1}$.

Let $b = b^r_{f_{t^0}}(S)$ be defined as in §2. (Here $r = \operatorname{corank} B^\ell_{f_{t^0}}(S)$.) Then we have:

(2) $b \geq d(x^0,D_\ell(f_{t^0}))^{\lambda_2}$

If $r>0$ is any constant, let $U_{(y^0,t^0)}(r)$ denote the neighbourhood of (y^0,t^0) in $R^p\times R$ which is the product of the two balls in R^p and R with center at y^0 and t^0 respectively, and with radius r. For each i we define the neighbourhood $U_{(x_i^0,t^0)}(r)$ in a similar way, and we put $U_S(r) = \bigcup_{i=1}^{\ell} U_{(x_i^0,t^0)}(r)$. Now let $r = (1/C)d(x^0,D_\ell(f))^{\lambda_3}$, $r' = (1/C')d(x^0,D_\ell(f_{t^0}))^{\lambda_3}$, and $r'' = (1/C')d(x^0,D_\ell(f_{t^0})^{\lambda_4}$. Then we have $r'/2>r''$, $r/2>r''$, $r'>r$, and r, r' and r'' satisfy the following:

(3) $F(U_S(r)) \subset U_{(y^0,t^0)}(r')$. Further, consider the germ f_{t^0}. Let U_S and U_{y^0} be the neighbourhoods of S and y^0, and I the interval at 0, that we get from applying Proposition 2.5 with f_{t^0} and h in the hypothesis. Then we have that $|t-t^0|<r'$ implies that $t-t^0\in I$. Further, we have that $U_S(r)\cap R^n\times\{t\} \subset U_S\times\{t\}$ and $U_{(y^0,t^0)}(r')\cap R^p\times\{t\} \subset U_{y^0}\times\{t\}$.

(4) The collection $\{U_{(y^0,t^0)}(r''/2):(y^0,t^0)\in\Lambda\}$ is a cover of $F(\Sigma(F))-\{0\}\times[0,1]$.

(5) For each $(x,t)\in F^{-1}(U_{(y^0,t^0)}(r''))\cap(\sim U_S(r/2))$ we have $J(f_t)(x)\geq\|x\|^{\lambda_5}$. (Here $J(f_t)$ is the sum of the squares of $Df_t(x)$.)

(6) The number of neighbourhoods $\{U_{(y,t)}(r'):(y,t)\in\Lambda\}$ intersecting each particular $U_{(y^0,t^0)}(r')$ in a common point is bounded by the constant $d(x^0,D_\ell(f_t))^{-\lambda_6}$.

Further, to get (4) and (5) to hold λ_3,λ_4 have to be chosen dependent of the particular (y^0,t^0) with $\lambda_4>\lambda_3$. Putting up, in advance, some lower bound for the λ_i's, we can choose Λ in such a way that all the corresponding λ_i's have this lower bound. Having chosen the lower bound, we can put up some common upper bound for all the λ_i's, independent of the particular $(y^0,t^0)\in\Lambda$, but dependent of the lower bound we already have chosen. Further, the upper bound will also be dependent of the constant and exponent of $L_1(f),\ldots,L_{p+1}(f)$, and of bounds of the derivatives of f and h up to some order in some neighbourhood of 0. The C and C' will only be dependent of bounds of the 1. order derivatives of f and h. Further, we can always choose $C,C'>1$.

The proof of Proposition 4.2 will be given in §5 and §6. The interest of Proposition 4.2 is that it will imply the following.

Theorem 4.3. Suppose $f:(R^n,0) \to (R^p,0)$ is a C^∞ map-germ satisfying $L_1(f),\ldots,L_{p+1}(f)$. Then for each k, $0 \leq k \leq \infty$ there exists an $m = m(k)$ such that $m < \infty$ if $k < \infty$ and $m = \infty$ if $k = \infty$, and such that f is $m(k)$-$\mathcal{A}^{(k)}$ determined.

Proof of Theorem 4.3. This proof will occupy the rest of this section. We will suppose that we already have proved 4.2. For each k, we want to find an $m(k)$ which satisfies the hypothesis of 4.1. Then the conclusion of 4.3 will follow from the conclusion of 4.1. If, for each k, $m(k)$ is chosen larger than the m in 4.2 (or 3.1), and h is such that $j^{m(k)}h(0) = 0$, then the corresponding F will possess a set Λ of "good points". Now we want to use that Λ satisfies (3) and (4) to construct the vectorfields X and Y of the hypothesis of 4.1 such that (1) of 4.1 is satisfied. Further, we will use that Λ satisfies (1), (2), (5) and (6) to show that if $m(k)$ is choosen sufficiently large (dependent on k), then also (2) of 4.1 will be satisfied. To show how X and Y can be constructed, let us first suppose that X and Y are vectorfields of the form described in 4.1. Define $\xi_t(x) = -\sum_{i=1}^{n} X_i(x,t)\frac{\partial}{\partial x_i}$, $\eta_t(y) = \sum_{i=1}^{p} Y_j(y,t)\frac{\partial}{\partial y_j}$. Then ξ_t, η_t can be considered as germs of t-parameter families of vectorfields at $R^n-\{0\}$, $R^p-\{0\}$ respectively. Let $h = (h_1,\ldots,h_p):(R^n,0) \to (R^p,0)$ be such that $j^m h(0) = 0$, where m is the m of 4.2 (or 3.1). Let for each t, $\tau_t(x)$ be the vectorfield along f_t defined by $\tau_t = \sum_{j=1}^{p} h_j \frac{\partial}{\partial y_j} \circ f_t$. (Here, f_t is defined by $f_t(x) = f(x)+th(x)$.)

Then it is easy to see that the condition (1) of 4.1 is equivalent with the condition

4.3.1 $$\tau_t = tf_t(\xi_t) + \omega f_t(\eta_t)$$

to hold for each $t \in [0,1]$ in a common punctured neighbourhood of 0.

To prove 4.3, it is therefore sufficient to find an $m(k)$ (for each k) such that if $j^{m(k)}h(0) = 0$, then we can find ξ_t, η_t satisfying 4.3.1, and such that the derivatives of ξ_t, η_t (also including the derivatives in the t-direction) satisfy inequalities which are analogous to those in (2) of 4.1.

To find such ξ_t, η_t, we will first construct ξ_t, η_t locally around $F(\Sigma(F))$, and $F^{-1}(F(\Sigma(F)))$, and then patch the local solutions carefully together by a partition of unity. Then we have to modify ξ_t away from $\Sigma(F)$. First let us state and prove a number of technical sublemmas which will contain most of the details we need for the construction of ξ_t and η_t. The first sublemma is the following:

<u>Sublemma 4.4</u>. Suppose that $g:(R^n,0) \to (R^p,0)$ is ∞-$\mathcal{K}$ determined. Then there exists a representative of g defined on some neighbourhood of 0 such that the following is satisfied: Let $\{x_1,\ldots,x_\ell\} \subset \Sigma(g)$ be such that $g(x_1) = \ldots = g(x_\ell) = y$. Then there exists $\lambda>0$ such that for all pair of indices i,j we have $\|x_i\| > \|x_j\|^\lambda$. Further, if $x \in \Sigma(g) - \{0\}$ and $y = f(x)$, then we also have $\|y\|^2 < \|x\|$ and $\|y\| > \|x\|^\lambda$.

<u>Proof of 4.4</u>. Since we know that g is ∞-$\mathcal{K}$ determined, there exists a constant $\lambda>0$ such that we have $\|g(x)\| + J(g)(x) \geq \|x\|^{\lambda-1}$ in a neighbourhood of $0 \in R^n$. Since each $x_i \in \Sigma(g)$, we get $\|g(x_i)\| \geq \|x_i\|^{\lambda-1}$ for each i. Since the derivatives of g are

bounded, we have that $\|g(x)\|<K\|x\|$ for some K bounding the derivatives of g. From this we deduce: $\|x_j\|^{\lambda-1}\leq\|g(x_j)\| = \|g(x_i)\|<K\|x_i\|$. Let us suppose that $\|x_j\|<\frac{1}{K}$. Then we deduce that $\|x_j\|^{\lambda}<\|x_i\|$. Note that λ is only dependent of the Lojasiewicz exponent in the Lojasiewicz inequality associated to ∞-$\mathcal{K}$ determinancy. Arguing as above, we have that $\|y\|<K\|x\|$. Let us suppose $\|y\|<1/K$. Then we deduce that $\|y\|^2<\|x\|^2$. Also, since $x\in\Sigma(g)$ and we may suppose that $\|x\|<1$, we get that $\|y\|\geq\|x\|^{\lambda-1}>\|x\|^{\lambda}$.

Now we have:

<u>Sublemma 4.5</u>. Let m be the integer of Proposition 4.2. Then, for each k such that $0\leq k\leq\infty$, there exists $\bar{m}(k)>m$ such that $\bar{m}(k)<\infty$ if $k<\infty$, and $\bar{m}(k) = \infty$ if $k = \infty$, and such that the following is satisfied:

Let $j^{\bar{m}(k)}h(0) = 0$, and let F and Λ be as in 4.2. (Λ exists since $\bar{m}(k)>m$.) Let also the notation be as in 4.2. Let $(y^0,t^0)\in\Lambda$, and consider the neighbourhood $U_{(y^0,t^0)}(r")$. For each t such that $\|t-t^0\|<r"$, put $U_t = U_{(y^0,t^0)}(r")\cap R^p\times\{t\}$. Then there exist vectorfields $\xi_t\in V(f_t^{-1}(U_t))$, $\eta_t\in V(U_t)$ which satisfy 4.3.1. Further, ξ_t,η_t are smooth also in the t parameter, and their derivatives satisfy inequalities which are analogous to (2) of 4.1.

<u>Proof of 4.5</u>. Let $\bar{m}>m$ be some integer. Let $j^{\bar{m}}h(0) = 0$, and let F and Λ be as in 4.2. Let $(y^0,t^0)\in\Lambda$, and let $S = \{x_1^0,\dots,x_\ell^0\} = f_{t^0}^{-1}(y^0)\cap\Sigma(f_{t^0})$. Consider $U_S(r)$ and $U_{(y^0,t^0)}(r')$ defined in Proposition 4.2. From 4.2 we have that each level-map f_t, $t\in[0,1]$, of F satisfies $L_1(f_t),\dots,L_{p+1}(f_t)$. From Lemma 3.4 follows that each f_t is ∞-$\mathcal{K}$ determined. From 4.4 we have that

4.5.1 $$\|x_i^0\| \geq \|x_j^0\|^{\lambda},$$

4.5.2 $$\|x_i^0\| > \|y^0\|^2,$$

4.5.3 $$\|y^0\| > \|x_i^0\|^{\lambda},$$

for each i,j for a suitable $\lambda>0$. Choosing $\bar{m}$ sufficiently large, it follows from the proof of 4.4 that λ is independent of the particular t^0. In fact, it is easy to see that λ only is dependent of the Lojasiewicz inequality that proves that $f = f_0$ is ∞-$\mathcal{K}$ determined. Now we have $U_S(r) = \bigcup_{i=1}^{\ell} U_{(x_i^0,t^0)}(r)$. Suppose that $x \in U_{(x_i^0,t^0)}(r)$. Then we have that $\|x\| \leq \|x_i^0\|+r = \|x_i^0\|+(1/C)d(x^0,D_\ell(f))^{\lambda_3}$. Since $d(x^0,D_\ell(f)) \leq \|x_i^0\|$, and we may suppose that $\|x_i^0\|<\frac{1}{2}$, $\|x\|<\frac{1}{2}$, $\lambda_3>2$, and $C>1$, we get that

4.5.4 $$\|x\|^2 \leq \|x_i^0\|.$$

We also get, using the opposite triangle inequality, that

4.5.5 $$\|x_i^0\|^2 \leq \|x\|.$$

Now we have $r' < d(x^0,D_\ell(f))^{\lambda_3} \leq \|x_i^0\|^{\lambda_3} \leq \|y^0\|^{\lambda_3/\lambda}$, where we have used 4.5.3 and that we can take $C'>1$. Suppose that $y \in U_{(y^0,t^0)}(r')$. Then we deduce that $\|y\| \leq \|y^0\|+\|y^0\|^{\lambda_3/\lambda}$. Since we by 4.2 can put up any lower bound for λ_i in advance, we may suppose that $\lambda_3/\lambda>2$. From this we can deduce that in a neighbourhood of 0 we have

4.5.6 $$\|y\|^2 \leq \|y^0\|,$$

4.5.7 $$\|y^0\|^2 \leq \|y\|.$$

Now, consider the right inverse σ_t of $tf_t+\omega f_t$ which, by Proposition 2.5, is defined on $V(f_t|U_S')$ for each t with $t-t^0 \in I$, and which takes values in $V(U_S) \times V(U_{y^0})$. (Here I, U_S, U_{y^0} are defined in 2.5.) Consider τ_t's restriction to U_S', and apply σ_t to this restriction. We obtain the vectorfields $(\xi_t,\eta_t)\in V(U_S)\times V(U_{y^0})$ such that $\tau_t|U_S = tf_t(\xi_t)+\omega f_t(\eta_t)$ for each t such that $t-t^0\in I$. Using 4.2 (3), it follows that for $|t-t^0|<r$ we can restrict ξ_t to $U_S(r)\cap R^n\times\{t\}$, and for $|t-t^0|<r'$ η_t can be restricted to $U_{y^0}(r')\cap R^p\times\{t\}$. Let us still denote these restrictions by ξ_t and η_t. By Proposition 2.5 we have the estimates

4.5.8 $$\|\xi_t\|_{k,U_S(r)} \leq C_{kf}(ab)^{-\beta k}\|\tau_t\|_{\gamma_k,U'_S}$$

4.5.9 $$\|\eta_t\|_{k,U_{y^0}(r')} \leq C_{kf}(ab)^{-\beta k}\|\tau_t\|_{\gamma_k,U'_S}.$$

In 4.5.8 and 4.5.9 we have $a = a^\ell_{f_{t^0}}(x^0)$, $b = b^r_{f_{t^0}}(S)$ where $r = \operatorname{corank} B^r_{f_{t^0}}(S)$. For m large enough $f_t = f+th$ satisfies the conclusion of 3.3, so

4.5.10 $$a = a^\ell_{f_{t^0}}(x^0) \geq \bar{C}_\ell d(x^0,D_\ell(f))^{\bar{\alpha}_\ell},$$

where $\bar{C}_\ell,\bar{\alpha}_\ell$ are independent of the particular t^0. Using (1) of 4.2, we get that

4.5.11 $$a \geq \bar{C}_\ell(\min_{1\leq i\leq\ell}\|x^0_i\|^{\bar{\alpha}_\ell\lambda_1}).$$

If we suppose that $(x,t)\in U_S(r)$, and use 4.5.1, 4.5.4 and 4.5.11, we get that

4.5.12 $$a \geq \bar{C}_\ell\|x\|^{2\bar{\alpha}_\ell\lambda_1\lambda}.$$

Using 4.2 (2) and the same kind of arguments, we get that

4.5.13 $$b \geq \|x\|^{2\lambda\lambda_1\lambda_2} \qquad \text{for } (x,t)\in U_S(r).$$

So we get that

4.5.14 $$ab \geq \bar{C}_\ell\|x\|^{2\lambda\lambda_1(\bar{\alpha}_\ell+\lambda_2)} \qquad \text{for } (x,t)\in U_S(r).$$

Now since $j^{\bar{m}}h(0) = 0$, it follows that the derivatives of τ_t of order equal or less than k are $o(\|x\|^{\bar{m}-k})$.

Now we may suppose that our map f is unstable at 0 (otherwise, everything we wish to prove are already well known). The map $a^{\ell}_{f_t}$ has therefore a zero at $0\in(R^n)^{\ell}$, and a Taylor expansion argument will show that $|a^{\ell}_{f_t}(x)|<K\|x\|$ for a suitable K. From the proof of 2.4 and 2.5, it is easy to see that the constant β, appearing in the definition of the radius of U_S, U'_S and U_{y^0} in 2.4 and 2.5, can be chosen to be larger than some upper bound we put up in advance. Putting up a sufficiently large lower bound for this β, and using that $|a^{\ell}_{f_{t^0}}(x^0)|<K\|x^0\|$, it will follow that the radius of U_S and U_{y^0} is smaller than $\min\limits_{1\leq i\leq \ell}\|x^0\|^2$ and $\|y^0\|^2$ respectively. Now if $x,x'\in U'_S$, we deduce from this and from 4.5.1, that $\|x\|\geq\|x'\|^{4\lambda}$ (provided we are sufficiently close to 0). Now using this, 4.5.8, 4.5.14, we deduce that the derivatives of ξ_t of order equal or less than k are $o(\|x\|^{m_k})$ where $m_k = ((\bar{m}-\gamma_k)/4\lambda)-\beta_k 2\lambda\lambda_1(\bar{\alpha}_\ell+\lambda_2)$. Choosing $\bar{m}>4\lambda(k+\beta_k 2\lambda\lambda_1(\bar{\alpha}_\ell+\lambda_2)+1)+\gamma_k$, we get that the derivatives are $o(\|x\|^{k+1})$. This choise is possible since the λ_i's have upper bounds independent of (y^0,t^0). Using 4.5.3, 4.5.4, and 4.5.7, we deduce that if $(x,t)\in U_S(r)$, $(y,t')\in U_S(r')$, then $\|y\|>\|x\|^{4\lambda}$. Using 4.5.9, and reasoning as above, we get that the derivatives of η_t of order k also $o(\|y\|^{m_k})$. We thus get that ξ_t,η_t satisfy inequalities similar to those in 4.1 if we are close to 0.

The ξ_t and η_t constructed above are defined on $U_S(r)$ and $U_{(y^0,t^0)}(r')$, and by 2.5 satisfy 4.3.1. Since we actually want to solve 4.3.1 on $F^{-1}(U_{(y^0,t^0)}(r''))$, we also need to have a solution valid on $F^{-1}(U_{(y^0,t^0)}(r''))\cap\sim(U_S(r))$. We therefore will need to extend and modify the above constructed ξ_t,η_t. To do so, we start by extending ξ_t,η_t on $R^n\times R$, and $R^p\times R$ respectively.

Let ρ,ρ' be two bump functions defined on $R^n\times R$, $R^p\times R$ respectively, such that $\rho\equiv 1$ on $U_S(3r/4)$, $\rho\equiv 0$ on $\sim U_s(r)$, and $\rho'\equiv 1$ on $U_{(y^0,t^0)}(r'/2)$, $\rho'\equiv 0$ on $\sim U_{(y^0,t^0)}(r')$. By 4.2 (3) and 2.5, $U_S(r)$ consists of a union of disjoint balls, so it is possible to define such a ρ. Put $\tilde{\xi}_t = (\rho' oF)\rho\xi_t$, and $\tilde{\eta}_t = \rho'\eta_t$. Now we can consider $\tilde{\xi}_t,\tilde{\eta}_t$ as defined on $R^n\times R$ and $R^p\times R$ respectively. Further, since by 4.2 $r'/2>r''$, we get that $\tilde{\xi}_t\equiv\xi_t$ on $F^{-1}(U_{(y^0,t^0)}(r''))\cap U_S(3r/4)$ and that $\tilde{\eta}_t\equiv\eta_t$ on $U_{(y^0,t^0)}(r'')$.

Let us estimate the derivatives of $\tilde{\xi}_t,\tilde{\eta}_t$. From the usual construction of "bump" funtions the derivatives of ρ and ρ' are bounded by negative powers of r and r'. Now, close to 0 we have $d(x^0,D_\ell(f))<1/C$, $d(x^0,D_\ell(f))<1/C'$. This implies that $r>d(x^0,D_\ell(f))^{\lambda_3+1}$, and that $r'>d(x^0,D_\ell(f))^{\lambda_3+1}$. Using 4.2 (1) and the fact that the λ_i's have upper bounds, we get that the derivatives of ρ,ρ' are bounded by negative powers of $\min_{1\leq i\leq\ell}\|x_i^0\|$. If $x\in U_S(r)$, it follows from 4.5.1 and 4.5.4 that $\|x\|^{2\lambda}\leq\min_{1\leq i\leq\ell}\|x_i^0\|$. So for any $x\in U_S(r)$, the derivatives of ρ on $U_S(r)$ or ρ' on $U_{(y^0,t^0)}(r')$ are bounded by negative powers of $\|x\|$, and these powers are not depending on the particular x. Since the derivatives of F is bounded, it is also clear that the derivatives of $\rho' oF\rho$ have similar bounds on $U_S(r)$. Since the derivatives of ξ_t of order k were $o(\|x\|^{m_k})$ with $m_k = ((\bar{m}-\gamma_k)/4\lambda)-2\beta_k\lambda\lambda_1(\bar{\alpha}_\ell+\lambda_2)$, it is clear that we can get the derivatives of $\tilde{\xi}_t$ of order less or equal k to be $o(\|x\|^{k+1})$ on $U_S(r)$ by choosing $\bar{m}$ sufficiently large. Since $\rho\equiv 0$ outside $U_S(r)$, the derivatives of $\tilde{\xi}_t$ are 0 outside $U_S(r)$.

Let $y\in U_{(y^0,t^0)}(r')$. Using 4.5.2, 4.5.6, (1) of 4.2 and that $r'>d(x^0,D_\ell(f))^{\lambda_3+1}$, we can deduce that $r'>\|y\|^{4\lambda_1(\lambda_3+1)}$. So for any $y\in U_{(y^0,t^0)}(r')$, the derivatives of ρ' are bounded by negative powers of $\|y\|$. Using that the derivatives of η_t are $o(\|y\|^{m_k})$, we can also obtain that the k'th order derivatives of $\tilde{\eta}_t$ are $o(\|y\|^{k+1})$ if $\bar{m}$ is large.

Next, put $\tilde{\tau}_t = tf_t(\tilde{\xi}_t)+\omega f_t(\eta_t)$. Since $\tau_t = tf_t(\xi_t)+\omega f_t(\eta_t)$ on $U_S(r)$, we get that $\tau_t\equiv\tilde{\tau}_t$ on $F^{-1}(U_{(y^0,t^0)}(r'/2))\cap U_S(3r/4)$. Now, for each $x\notin\Sigma(f_t)$, consider the matrix
$I_t(x) = (Df_t(x))^t((Df_t(x))(Df_t(x))^t)^{-1}$. (Here $(Df_t(x))^t$ is the transpose of $Df_t(x)$.) Define $\tilde{\tilde{\eta}}_t(x) = I_t(x)(\tau_t(x)-\tilde{\tau}_t(x))$. From 4.2 (5) follows that $(x,t)\in F^{-1}(U_{(y^0,t^0)}(r'')\cap(\sim U_S(r/2))$ implies that $x\notin\Sigma(f_t)$, so $\tilde{\tilde{\xi}}_t$ is defined on $F^{-1}(U_{(y^0,t^0)}(r''))\cap(\sim U_S(r/2)$. On the other hand, since $\tau_t-\tilde{\tau}_t\equiv 0$ on $F^{-1}(U_{(y^0,t^0)}(r'/2))\cap U_S(3r/4)$ and $U_{(y^0,t^0)}(r'/2)\supset U_{(y^0,t^0)}(r'')$ (since $r''>r'/2$), we can extend ξ_t to a smooth vector field on $F^{-1}(U_{(y^0,t^0)}(r''))$ putting $\tilde{\tilde{\xi}}_t\equiv 0$ on $F^{-1}(U_{(y^0,t^0)}(r'')\cap U_S(r/2)$. At last let us redefine ξ_t,η_t putting $\xi_t = \tilde{\xi}_t+\tilde{\tilde{\xi}}_t$, $\eta_t = \tilde{\eta}_t$. Then we will get that

$$\tau_t = tf_t(\xi_t) + \omega f_t(\eta_t) \quad \text{on} \quad F^{-1}(U_{(y^0,t^0)}(r'')).$$

At last, we need to see that we can get the derivatives of ξ_t,η_t to satisfy inequalities similar to those in 4.1. We have already seen that this was the case for $\tilde{\xi}_t,\tilde{\eta}_t$. Now,
$\tilde{\tilde{\xi}}_t = I_t(x)(\tau_t(x)-\tilde{\tau}_t(x))$ where $\tilde{\tau}_t(x) = tf_t(\tilde{\xi}_t)+\omega f_t(\tilde{\eta}_t)$. We had that we can make the derivatives of order k of $\tilde{\xi}_t$ and $\tilde{\eta}_t$ to

be $o(\|x\|^{k+1})$ and $o(\|y\|^{k+1})$ on $R^n\times R$, $R^p\times R$ respectively by choosing $\bar{m}$ sufficiently large. (Recall that $\tilde{\xi}_t, \tilde{\eta}_t$ are 0 outside $U_S(3r/4)$ and $U_{(y^0,t^0)}(r')$). Since $f_t(x) = y$ implies that $\|y\|^2 < \|x\|$ in a neighbourhood of 0, and the derivatives of f_t are bounded, we can obtain that the derivatives of $\omega f_t(\tilde{\eta}_t)$ of order k also are $o(\|x\|^{k+1})$ (by choosing $\bar{m}$ somewhat larger if necessary). This implies that the derivatives of $\tilde{\tau}_t$ of order k will be $o(\|x\|^{k+1})$. Now, the entries of $I_t(x)$ are bounded by derivatives of f_t divided by $J(f_t)(x)$, (note that $Jf_t(x) = \det(Df_t(x)Df_t(x)^t)$), and by 4.2 (5) $J(f_t)(x) \geq \|x\|^{\lambda_5}$ on $F^{-1}(U_{(y^0,t^0)}(r''))\cap(\sim U_S(r/2))$ and also $\tilde{\tilde{\xi}}_t \equiv 0$ on $F^{-1}(U_{(y^0,t^0)}(r''))\cap(U_S(r/2))$. Using the formula for the derivatives of a quotient, and choosing $\bar{m}$ even larger, we can also get that the derivatives of $\tilde{\tilde{\xi}}_t$ of order less or equal k are $o(\|x\|^{k+1})$. In this way we can deduce that ξ_t, η_t satisfy the desired inequalities. Note that in the above arguments we have to use that the λ_i's have upper bounds independent of (y^0,t^0) so the choise of $\bar{m}$ is independent of (y^0,t^0). In the case when $\bar{m} = \infty$, the above arguments will show that all the derivatives of ξ_t, η_t will be $o(\|x\|^k)$, $o(\|y\|^k)$ for any k. This shows that the derivatives of ξ_t, η_t satisfy inequalities analogous to 4.1 (2). This completes the proof of the sublemma.

Our next sublemma will be the following.

Sublemma 4.6. Let f, F, Λ, and all the remaining notation be as in 4.2. Let $V = \sim \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((3/4)r'')$, and put $U = F^{-1}(V) - \{0\}\times[0,1]$. For each $t\in[0,1]$ put $U_t = U\cap R^p\times\{t\}$. Then we have a map $\sigma_t : V(f_t|U_t) \to V(U_t)$, which is a right inverse to the map $tf_t : V(U_t) \to V(f_t|U_t)$. Further σ_t satisfies the following condition:

For each non-negative integer k we have a non-negative integer γ_k, and positive constants α_k, K_k satisfying the following: If $\tau_t \in V(f_t|U_t)$, $\xi_t = \sigma_t(\tau_t)$, and $x_0 \in U_t$, then we have

$$(1) \qquad \sup_{|\beta|\leq k} \left\| \frac{\partial^{|\beta|}\xi_t}{\partial^\beta x}(x_0) \right\| \leq K_k \|x_0\|^{-\alpha_k} \sup_{|\beta|\leq\gamma_k} \left\| \frac{\partial^{|\beta|}}{\partial^\beta x} \tau_t(x_0) \right\|.$$

If τ_t varies smoothly in t, then ξ_t is also smooth in t, and also the derivatives of ξ_t which involves the t-direction satisfy estimates of type (1). The constants α_k, K_k will be independent the particular t.

Proof of 4.6. Let $(x,u)\in U$, and put $I_t(x) = Df_t(x)^t(Df_t(x)(Df_t(x))^t)^{-1}$. Since it follows from 4.2 (4) that U does not contain any singular points of F, $I_t(x)$ exists. Define $\sigma_t(\tau_t) = \xi_t$ by $\xi_t(x) = I_t(x)\tau_t(x)$. Reasoning as in the proof of 4.5, it is clear that (1) follows if we can show that $J(f_t)(x) \geq \|x\|^\alpha$ for a suitable α independent of t whenever $x\in U$. To prove such an inequality, let $x\in U$, and let x' be a point in $\Sigma(f_t)$ such that $\|x-x'\| = d(x,\Sigma(f_t))$. (Such a point always exists if x is sufficiently close to 0.)

Then Lemma 3.10 implies that $J(f_t)(x) \geq \|x-x'\|^{\alpha'}$ where α' can be chosen to be independent of t and x. Let us first suppose that

4.6.1 $$\|x-x'\| \geq \|x\|^2.$$

Then we get that

4.6.2 $$J(f_t)(x) \geq \|x\|^{2\alpha'}.$$

Next suppose that

4.6.3 $$\|x-x'\| < \|x\|^2.$$

Povided $\|x\|<\frac{1}{2}$, we deduce that

4.6.4 $$\|x'\| > \|x\|^2.$$

Put $y' = f_t(x')$. Since each f_t is ∞-$\mathcal{K}$ determined (Lemma 3.4), we must have $f_t^{-1}(0)\cap\Sigma(f_t) = \{0\}$ in a common neighbourhood of 0. Since $x\neq 0$, 4.6.4 implies that $x'\neq 0$, and since $x'\in\Sigma(f_t)$ we get $y'\neq 0$. By 4.2 (4) $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r''/2)$ is a cover of $F(\Sigma(F))-\{0\}\times[0,1]$. This implies that we can find $(y^0,t^0)\in\Lambda$ such that

4.6.5 $$\|(y',t)-(y^0,t^0)\| < r''/2.$$

Let $y = f_t(x)$. Since $(y,t)\notin \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((3/4)r'')$, we get that

4.6.6 $$\|y-y'\| \geq (3/4)r'' - r''/2 = r''/4.$$

Let $f_{t^0}^{-1}(y^0)\cap\Sigma(f_t) = \{x_1^0,\ldots,x_\ell^0\}$. Reasoning as in 4.4, we get that

4.6.7 $$\|y^0\|^2 \leq \|x_i^0\| \quad \text{for each } i.$$

Using 4.2 (1) and the definition of r'', we deduce that

4.6.8 $$r'' \geq (1/C')\|y^0\|^{2\lambda_1\lambda_4} \geq \|y^0\|^{2\lambda_1\lambda_4+1}$$

(provided $\|y^0\|<1/C'$). We thus get that

4.6.9 $$\|y-y'\| \geq (1/4)\|y^0\|^{2\lambda_1\lambda_4+1}.$$

On the other hand reasoning as in 4.5.6, we get that

4.6.10 $$\|y^0\| \geq \|y'\|^2.$$

So we get from this, and from 4.6.9 that

4.6.11 $$\|y-y'\| \geq (1/4)\|y'\|^{4\lambda_1\lambda_4+2} \geq \|y'\|^{4\lambda_1\lambda_4+3}.$$

(We suppose that $\|y'\|<1/4$.) Since the derivatives of f_t are bounded, we must have that

4.6.12 $$\|y-y'\|^2 \leq \|x-x'\|$$

in a neighbourhood of 0. Using this, 4.6.11, and 4.4, we get that

4.6.13 $$\|x-x'\| \geq \|x'\|^{2\lambda(4\lambda_1\lambda_4+3)},$$

where λ is the λ of Sublemma 4.4. Using 4.6.4 we get that

4.6.14 $$J(f_t)(x) \geq \|x\|^{4\lambda(4\lambda_1\lambda_4+3)\alpha'}$$

So in any case we have that

$$J(f_t)(x) \geq \|x\|^{\alpha}$$

for a suitable α. To get that α independent of the point (x,t), we have to use that the λ_i's all have upper bounds. This completes the proof of the sublemma.

Sublemma 4.7. Let f, F, Λ, and all the remaining notation be as in 4.2. For each $(y^0,t^0)\in\Lambda$, let r_1,r_2 be given constants such that $r'>qr_1>r_1>sr_2>r_2>r''/2$, where q and s are fixed constants >1. Consider $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)$, and $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_2)$. Let $(y,t)\in \overline{\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_2)}-\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_2)$, and suppose that $\|y\|\neq 0$. Then $(y,t)\in \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)$.

<u>Proof</u>. Let (y_n,t_n) be a sequence in $\bigcup_{(y^0t^0)\in\Lambda} U_{(y^0,t^0)}(r_2)$ such that $(y_n,t_n) \to (y,t)$ when $n\to\infty$. Let, for each n, $(y_n,t_n)\in U_{(y_n^0,t_n^0)}(r_2)$, where $(y_n^0,t_n^0)\in\Lambda$. Then we have that

4.7.1 $$\|(y_n,y_n)-(y_n^0,t_n^0)\| < r_2.$$

Since $(y_n,t_n)\in U_{(y_n^0,t_n^0)}(r')$, we can reason as in 4.5.6, and deduce that

4.7.2 $$\|y_n\|^2 < \|y_n^0\|.$$

Since $y_n\to y$ and $\|y\|\neq 0$, it follows that $\|y_n\|$ and $\|y_n^0\|$ are bounded away from 0. Let $S_n = f_{t_n^0}^{-1}(y_n^0)\cap\Sigma(f_{t_n^0}) = \{x_{n1}^0,\ldots,x_{n\ell_n}^0\}$. From 4.4 we deduce that $\|y_n^0\|^2\leqslant\|x_{ni}^0\|$ for $i = 1,\ldots,\ell_n$, and from 4.2 (1) and the fact that $r_2>r''/2$, we can deduce that

4.7.3 $$r_2 \geqslant (1/2C')\|y_n^0\|^{2\lambda_1\lambda_4}.$$

Since λ_1,λ_4 have upper bounds, it follows that r_2 is bounded away from 0 when n varies. So let $B>0$ be some lower bound of all the r_2's associated to $\{(y_n^0,t_n^0)\}$, and choose n so large that $\|(y,t)-(y_n,t_n)\|<(s-1)B$. Then we deduce that

4.7.4 $$\|(y,t)-(y_n^0,t_n^0)\| < (s-1)B+r_2 \leqslant (s-1)r_2+r_2 \leqslant r_1.$$

So we get that $(y,t)\in U_{(y_n^0,t_n^0)}(r_1)$. Since $(y_n^0,t_n^0)\in\Lambda$, the sublemma is proved.

Our next sublemma is:

Sublemma 4.8. Let the notation be as in 4.7. Then there exists a C^∞ function Φ on $R^p \times R - \{0\} \times R$ with values in $[0,1]$ such that $\phi \equiv 1$ on $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_2)$ and $\rho \equiv 0$ on $\sim \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)$.

Further, for every positive integer k, there exists a positive integer γ_k, and a neighbourhood of $\{0\}\times[0,1]$ in $R^p \times R$ such that each derivative of Φ of order less or equal k at a point (y,t) in this neighbourhood is bounded by $\|y\|^{-\gamma_k}$. The γ_k are only dependent on the upper bounds of the λ_i's of 4.2, and on s of 4.7.

Proof of 4.8. For each $(y^0,t^0)\in\Lambda$, define a bump function $\rho = \rho^{(y^0,t^0)}$ on $R^p \times R$ such that $\rho^{(y^0,t^0)} \equiv 1$ on $U_{(y^0,t^0)}(r_2)$, and $\rho^{(y^0,t^0)} \equiv 0$ on $\sim U_{(y^0,t^0)}(r_1)$. Since $r_1 > sr_2$, we have by the usual construction of such bump functions that the derivatives of $\rho^{(y^0,t^0)}$ are bounded by negative powers of r_2. (These powers will in this case also depend on s.) From inequality 4.7.3, it follows that the derivatives can be bounded by negative powers of $\|y^0\|$, and from 4.7.2 (or 4.5.6), follows that the derivatives at a point (y,t) are bounded by negative powers of $\|y\|$. (Recall that $\rho \equiv 0$ outside $U_{(y^0,t^0)}(r_1) \subset U_{(y^0,t^0)}(r')$.)

Put $g = \sum_{(y^0,t^0)\in\Lambda} \rho^{(y^0,t^0)}$. We want to show that g is C^∞ on $R^p \times R - \{0\}\times R$. Since $r' > r_1$, it follows from 4.2 (6) that the sum defining g is locally finite in $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)$, so g is C^∞ on this set. From the proof of 4.7 follows that $\overline{\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)} - \{0\}\times R$ is contained in $\bigcup_{(y^0,t^0)\in\Lambda} U_{(t^0,t^0)}(r')$. (We get this by replacing r_2 and r_1 with r_1 and r' in the proof of 4.7.) So, by 4.2 (6), and the fact that each $\rho^{(y^0,t^0)}$ is

0 outside $\sim \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)$, it follows that g actually is a locally finite sum at all points. So g is C^∞. Now, the number of non-vanishing terms at points in $U_{(y^0,t^0)}(r')$ is bounded by $d(x^0,D_\ell(f))^{-\lambda_6}$ (4.2 (6)).

From several inequalities in 4.5 and from 4.2 (1) it follows that $d(x^0,D_\ell(f)) \geq \|y^0\|^{2\lambda_1}$, and since $(y,t)\in U_{(y^0,t^0)}(r')$ implies that $\|y^0\| > \|y\|^2$, we get that the number of non-vanishing terms at (y,t) are bounded by negative powers of $\|y\|$. This implies that the derivatives of g are bounded by negative powers of y. At last, let $\ell: R \to [0,1]$ be C^∞ such that $\ell(t)\equiv 0$ for $t\leq 0$ and $\ell(t)\equiv 1$ for $t\geq 1$. Put $\Phi = \ell o g$. Then $\Phi\equiv 1$ on $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_2)$, and $\Phi\equiv 0$ on $\sim \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}(r_1)$. Since the derivatives of ℓ are bounded, the derivatives of Φ will satisfy the desired estimates. This completes the proof of the sublemma.

Here is another sublemma which is similar to 4.8.

<u>Sublemma 4.9.</u> Let r_1, r_2 be such that $r > qr_1 > r_1 > sr_2 > r/2$ for $q,s>1$, and let the remaining notation be as in 4.2. Consider the collection of all sets S such that $S = f_{t^0}^{-1}(y^0) \cap \Sigma(f_t)$ for $(y^0,t^0)\in\Lambda$. Then we have a C^∞ function Φ defined on $R^n\times R - \{0\}\times R$, and with values in $[0,1]$, which is equal 1 on $\bigcup_S U_S(r_2)$, and which is 0 outside $\bigcup_S U_S(r_1)$. Further, the derivatives of Φ at (x,t) are bounded by negative powers of $\|x\|$ in a neighbourhood of $0\times[0,1]$ in a way similar to the Φ of 4.8.

<u>Proof</u>. For each S define a bump function ρ^S which is $\equiv 1$ on $U_S(r_2)$, and which is 0 outside $U_S(r_1)$. For $x\in U_S(r_1)$ it follows from several inequalities in 4.5 and from 4.2 (1) that

$$r_1 \geq (1/C)\|x\|^{2\lambda\lambda_1\lambda_3}.$$

(Here λ is the λ of Sublemma 4.4.) So the derivatives of ρ^S are by the usual construction bounded by negative powers of $\|x\|$. Put $g = \sum_S \rho^S$. If we argue as in 4.7, we get that $\overline{\bigcup_S U_S(r_1)} - \{0\}\times R \subseteq \bigcup_S U_S(r)$. Since $F(U_S(r)) \subset U_{(y^0,t^0)}(r')$, it follows from 4.2 (6) that the union $\bigcup_S U_S(r)$ actually is locally finite, and that the number of neighbourhoods that can intersect satisfies the bound in 4.2 (6). Now we can use the same sort of arguments as those in 4.8 to construct Φ and complete the proof of the sublemma.

After these preparatorious sublemmas, let us complete the proof of 4.3. Let $0 \leq k \leq \infty$, and let $\bar{m} \geq m$ where m is the m of 4.2. Let h be such that $j^{\bar{m}}h(0) = 0$, and let F, Λ and the remaining notation be as in 4.2. For each $(y^0,t^0)\in\Lambda$ we use 4.5 to construct vectorfields $\xi_t^{(y^0,t^0)}$, and $\eta_t^{(y^0,t^0)}$ such that $\tau_t = tf_t(\xi_t^{(y^0,t^0)}) + \omega f_t(\eta_t^{(y^0,t^0)})$ on $F^{-1}(U_{(y^0,t^0)}(r''))$. Choosing $\bar{m}$ sufficiently large we can by 4.5 get that the derivatives of $\xi_t^{(y^0,t^0)}$, $\eta_t^{(t^0,t^0)}$ up to order k are $o(\|x\|^{k+1})$ and $o(\|y\|^{k+1})$ respectively. (Here $k<\infty$, if $k = \infty$ and $\bar{m} = \infty$, they are $o(\|x\|^{\ell})$, $o(\|y\|^{\ell})$ for any ℓ.)

For each $(y^0,t^0)\in\Lambda$ find a C^∞ function $\rho^{(y^0,t^0)}$, with values in $[0,1]$, such that $\rho^{(y^0,t^0)} \equiv 1$ on $U_{(y^0,t^0)}((13/16)r'')$, and $\rho^{(y^0,t^0)} \equiv 0$ outside $U_{(y^0,t^0)}((15/16)r'')$. Put $g = \sum_{(y^0,t^0)\in\Lambda} \rho^{(y^0,t^0)}$. Arguing as in the proof of 4.8, we have that g is C^∞ on $R^p\times R - \{0\}\times R$, and that the derivatives of g are bounded by negative powers of $\|y\|$. Use 4.8 to find a C^∞

function Φ' such that $\Phi'\equiv 1$ on $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((49/64)r'')$, and $\Phi'\equiv 0$ outside $\sim \bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((51/64)r'')$.

Put $\tilde{\rho}^{(y^0,t^0)} = (\Phi'\rho^{(y^0,t^0)})/g$. Using 4.7 and that $g>1$ on $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((52/64)r'')$, it follows that $\tilde{\rho}^{(y^0,t^0)}$ is C^∞ on $R^p\times R-\{0\}\times R$, and we have that $\sum_{(y^0,t^0)\in\Lambda} \tilde{\rho}^{(y^0,t^0)}\equiv 1$ on $\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((49/64)r'')$. We also get from 4.8 that the derivatives of $\tilde{\rho}^{(y^0,t^0)}$ are bounded by negative powers of $\|y\|$.

Put $\tilde{\xi}_t = \sum_{(y^0,t^0)\in\Lambda} \tilde{\rho}^{(y^0,t^0)} \circ F\xi_t^{(y^0,t^0)}$, $\tilde{\eta}_t = \sum_{(y^0,t^0)\in\Lambda} \tilde{\rho}^{(y^0,t^0)}\eta_t^{(y^0,t^0)}$, and $\tilde{\tau}_t = tf_t(\tilde{\xi}_t)+\omega f_t(\tilde{\eta}_t)$. Note that $\tilde{\eta}_t, \tilde{\xi}_t, \tilde{\tau}_t$ are germs at $\{0\}\times[0,1]$ defined on $R^p\times R-\{0\}\times R$, and on $R^n\times R-\{0\}\times R$ respectively. By construction, we also get that $\tau_t\equiv\tilde{\tau}_t$ on $F^{-1}(\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((49/64)r''))\cap R^n\times[0,1]-\{0\}\times[0,1]$. Let us use Sublemma 4.6 to find a germ $\tilde{\tilde{\xi}}_t$ such that $tf_t(\tilde{\tilde{\xi}}_t) = \tau_t-\tilde{\tau}_t$ on $\sim F^{-1}(\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((3/4)r''))$. Since $\tau_t-\tilde{\tau}_t\equiv 0$ on $F^{-1}(\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((49/64)r''))$ and $49/64>3/4$, we will get from the proof of 4.6 that $\tilde{\tilde{\xi}}_t$ can be extended to a C^∞ vector-field also on $F^{-1}(\bigcup_{(y^0,t^0)\in\Lambda} U_{(y^0,t^0)}((3/4)r''))$, by putting $\tilde{\tilde{\xi}}_t\equiv 0$ on this set.

Let us define $\tau_t = \tilde{\xi}_t+\tilde{\tilde{\xi}}_t$ and $\eta_t = \tilde{\eta}_t$. We get that

$$\tau_t = tf_t(\xi_t) + \omega f_t(\eta_t).$$

Let us estimate the derivatives of ξ_t and η_t. Using (6) of 4.2 the estimates we had on $\eta_t^{(y^0,t^0)}$ and on $\tilde{\rho}^{(y^0,t^0)}$, we will get that we can obtain that the deivatives of $\tilde{\eta}_t$ of order less or equal to k are $o(\|y\|^{k+1})$ provided $\bar{m}$ is sufficiently large. (If $k = \infty$ and $\bar{m} = \infty$ they are $o(\|y\|^{\ell})$ for any ℓ.)

Now, also using (6) of 4.2, we see that the number of non-vanishing terms in the sum defining ξ_t are bounded by a negative powers of $\|y\|$. (Because $d(x^0, D_\ell(f)) > \|y^0\|^{2\lambda_1} \geq \|y\|^{4\lambda_1}$ for $y \in U_{(y^0,t^0)}(r)$.) Also the derivative of $\tilde{\rho}^{(y^0,t^0)}$ are bounded by such negative powers. Although we also have that the derivatives of $\xi_t^{(y^0,t^0)}$ up to order k can be made $o(\|x\|^{k+1})$, it might happen that we have points (x,t) such that $\|x\|$ is large compared to $\|f_t(x)\|$. This may prevent the derivatives of $\tilde{\rho}^{(y^0,t^0)} \circ F \xi_t^{(y^0,t^0)}$ to be $o(\|x\|^{k+1})$ for such points. When $(x,t) \in U_S(r)$, $y = f_t(x)$, it is proved in the proof of 4.5 that $\|y\| \geq \|x\|^{4\lambda}$ where λ is the λ of 4.4. Using this, we can prove that we can get the derivatives of ξ_t of order less than k to be $o(\|x\|^{k+1})$, provided $\bar{m}$ is large, and $(x,t) \in U_S(r)$ for some $S = \{x_1^0, \dots, x_\ell^0\} = f_{t^0}^{-1}(y^0) \cap \Sigma(f_{t^0})$, where $(y^0,t^0) \in \Lambda$. We thus only have to modify our solution ξ_t on $\sim \bigcup_S U_S(r)$.

Let $\gamma > 0$, and consider the neighbourhood $V(\Sigma(F),\gamma) = \bigcup_{x \in \Sigma(f_t) - \{0\}} \{(x',t') \mid \|(x',t')-(x,t)\| < \|x\|^\gamma\}$. We will first show that there exists a γ such that $V(\Sigma(F),\gamma) \subset \bigcup_S U_S(r/2)$. Let $\gamma > 0$, let $(x',t') \in V(\Sigma(F),\gamma)$, and let (x,t) be a point in $\Sigma(f_t)$ such that $\|(x',t')-(x,t)\| \leq \|x\|^\gamma$.

Now, $y = f_t(x)$ is a singular value, and by 4.2 (4), there exists $(y^0,t^0) \in \Lambda$ such that $\|(y^0,t^0)-(y,t)\| < r''/2$. From this follows easily that $\|y\|^2 < \|y^0\|$, and from 4.4 follows that $\|x\|^\lambda < \|y\|$. Put $y' = f_{t'}(x')$. In a neighbourhood of 0 we have that

$$\|(y,t)-(y',t')\|^2 \leq \|(x,t)-(x',t')\| < \|x\|^\gamma < \|y\|^{\gamma/\lambda} < \|y^0\|^{\gamma/2\lambda}.$$

So we get that $\|(y^0,t^0)-(y',t')\| \leq r''/2 + \|y^0\|^{\gamma/4\lambda}$. If $\|y^0\|$ is close to 0, we can deduce that $r''/2 \geq \|y^0\|^{2\lambda_1\lambda_4+1}$. (Using 4.2 (1) the definition of r'' and 4.4.) (See also the proof of 4.6.8.) So if we have that $\gamma > 4\lambda(2\lambda_1\lambda_4\lambda+1)$ we get that $\|(y^0,t^0)-(y',t')\| \leq r''$. (This is possible since λ_1, λ_4 have upper bounds independent of (y^0,t^0).) So we get $(y',t') \in U_{(y^0,t^0)}(r'')$.

Now, $\|(x',t')-(x,t)\| \leq \|x\|^\gamma$ implies that $\|x'\| \geq \|x\|^2$. Taylors formula will also imply that $(J(f_{t'})(x'))^2 \leq \|(x',t')-(x,t)\| \leq \|x\|^\gamma$. So we get that $J(f_{t'})(x') \leq \|x'\|^{\gamma/4}$. Now, if also γ is chosen such that $\gamma > 4\lambda_5$, $(x',t') \in \sim U_S(r/2)$ will contradict 4.2 (5). So we get that $V(\Sigma(f),\gamma) \subset \bigcup_S U_S(r/2)$ for $\gamma \geq \max(4\lambda(2\lambda_1\lambda_4\lambda+1), 4\lambda_5)$. (Note that such a choise is possible since the λ_i's have upper bounds.)

Next suppose that $(x',t') \in \sim V(\Sigma(F),\gamma)$. Let (x,t') be point such that $x \in \Sigma(f_{t'})$, and such that $\|x-x'\| = d(x, \Sigma(f_{t'}))$. Then we get that $\|x-x'\| \geq \|x\|^\gamma$. By Lemma 3.10 and Remark 3.11, we have that $J(f_{t'})(x') \geq \|x-x'\|^{\alpha'}$. If $\|x\| < \|x'\|^2$, we get that $\|x-x'\| \geq \|x'\| - \|x\| \geq \|x'\| - \|x'\|^2 \geq \frac{1}{2}\|x'\| \geq \|x'\|^2$. (We suppose that $\|x'\| < \frac{1}{2}$.) This implies that $J(f_{t'})(x') \geq \|x'\|^{2\alpha'}$. If $\|x\| \geq \|x'\|^2$, we have that $\|x-x'\| \geq \|x'\|^{2\gamma}$, and we get that $J(f_t(x') \geq \|x'\|^{2\gamma\alpha'}$. So in any case we get that $J(f_{t'})(x') \geq \|x'\|^{\hat{\gamma}}$ for $(x',t') \in \sim V(\Sigma(F),\gamma)$ for a suitable choise of $\hat{\gamma}$.

On $\sim V(\Sigma(F),\gamma)$ define $I_t(x) = (Df_t(x))^t(Df_t(x)(Df_t(x))^t)^{-1}$. Let, by Sublemma 4.9, Φ be a function with values in $[0,1]$ such that $\Phi \equiv 1$ on $\bigcup_S U_S((5/8)r)$ and $\Phi \equiv 0$ outside $\bigcup_S (U_S((7/8)r)$. Put $\hat{\xi}_t = \Phi\xi_t$. Now $\hat{\xi}_t \equiv 0$ outside $\bigcup_S U_S((7/8)r)$, and we had that we can get the derivatives of ξ_t of order k to be $o(\|x\|^{k+1})$ on $\bigcup_S U_S((7/8)r) \subset \bigcup_S U_S(r)$ if $\bar{m}$ is sufficiently large. Since the derivatives of Φ is bounded by negative powers of $\|x\|$ (use

4.9), we get that by choosing $\bar{m}$ even larger we can get the derivatives of $\hat{\xi}_t$ or order $\leq k$ to be $o(\|x\|^{k+1})$ everywhere. In the case when $k = \infty$ it will also follows from these arguments that all the derivatives of ξ_t will be $o(\|x\|^{\ell})$ for all ℓ provided $\bar{m} = \infty$.

On $\sim V(\Sigma(F),\gamma)$ put

$\hat{\hat{\xi}}_t(x) = I_t(x)(\tau_t(x)-tf_t(\hat{\xi}_t(x))-\omega f_t(\eta_t)(x))$. Since $\hat{\xi}_t \equiv \xi_t$ on $\bigcup_S U_S((5/8)r)$, we have that $\tau_t-tf_t(\hat{\xi}_t)-\omega f_t(\eta_t)\equiv 0$ on $\bigcup_S U_S((5/8)r)$. Since $V(\Sigma(f),\gamma) \subset \bigcup_S U_S(r/2)$, we get that $V(\Sigma(f),\gamma) \subset \bigcup_S U_S((5/8)r)$. It follows that $\hat{\hat{\xi}}_t$ can be extended to be C^{∞} outside $\{0\}\times R$ in a neighbourhood of $\{0\}\times[0]$, by putting $\hat{\hat{\xi}}_t \equiv 0$ on $V(\Sigma(f),\gamma)$.

Now we calculate that

$$\tau_t = tf_t(\hat{\xi}_t+\hat{\hat{\xi}}_t) + \omega f_t(\eta_t)$$

holds outside $\{0\}\times R$ in a neighbourhood of $\{0\}\times[0,1] \subset R^n\times R$.

Let us estimate the derivatives of $\hat{\hat{\xi}}_t$. From what we already have said about the derivatives of τ_t, $\hat{\xi}_t$, η_t, and from the fact that the derivatives of f is bounded, it follows that the derivatives of $\tau_t-tf_t(\hat{\xi}_t)-\omega f_t(\eta_t)$ of order $\leq k$ will be $o(\|x\|^{k+1})$ if $\bar{m}$ is sufficiently large. Since $\det(Df_t(x)Df_t(x)^t) = J(f_t)(x) \geq \|x\|^{\hat{\gamma}}$ on $\sim V(\Sigma(F),\gamma)$, and the entries of $Df_t(x)$ and $Df_t(x)^t$ are bounded by bounds of the derivatives of f_t, it follows the entries of $I_t(x)$ are bounded by some constants divided by $\|x\|^{\hat{\gamma}}$. From this it is easy to see that the derivatives of $\hat{\hat{\xi}}_t$ of order $\leq k$ can be made $o(\|x\|^{k+1})$ if $\bar{m}$ is chosen even larger. If $k = \infty$, and $\bar{m} = \infty$ all derivatives will be $o(\|x\|^{\ell})$ for all ℓ.

Putting $\xi_t = \hat{\xi}_t+\hat{\hat{\xi}}_t$, ξ_t,η_t will solve 4.3.1, and satisfy inequalities similar to (2) of 4.1. This completes the proof of 4.3.

§5. Approximation lemmas

In this section we will prove four approximation lemmas which will be necessary in the proof of Proposition 4.2. We want to show that it is possible to approximate singular values in the target with points satisfying (1), (2) and (5) of Proposition 4.2. In fact, this will be shown in Lemma 5.5. The other lemmas in this section can be regarded as "nested" sublemmas, which we will need in the proof of 5.5. To be more precise; Lemma 5.4 is only used in the proof of 5.5, Lemma 5.3 is used in the proof of 5.4 and so on. The statements in these lemmas will be rather technical. The first lemma is the following:

Lemma 5.1. Suppose $f:(R^n,0) \to R^p,0)$ satisfies $L_1(f),\dots,L_{p+1}(f)$. Let t be a positive integer, and let g be a positive polynomial in one variable. Then there exists a representative of f defined on some neighbourhood U of 0 such that the following holds:

Let $\nu>1$. Then there exists a positive polynomial $\beta'(\nu)$ such that if $x = (x_1,\dots,x_\ell)\in(U)^\ell$, $\ell\leqslant p$, and $\sum\|f(x_i)-f(x_j)\|^2 + \sum_{i=1}^{\ell} J(f)(x_i)<d(x,D_\ell(f))^{\beta'(\nu)}$, then the following is satisfied:

There exists a point $x' = (x_1',\dots,x_\ell')\in(U)^\ell$ with $x_i'\in\Sigma(f)$, and $f(x_i') = f(x_j')$ for all i,j, and there exists a positive polynomial $\beta(\nu)$ such that the following two conditions are satisfied:

(1) $\|x-x'\|\leqslant d(x',D_\ell(f))^{\nu^m\beta(\nu)}$ where $m = t \deg \beta$.

(2) Put $S' = \{x_1',\dots,x_\ell'\}$, and let $r = \operatorname{corank} B_f^\ell(S')$, where $B_f^\ell(S')$ is defined in §2. Put $b = b_f^r(S')$. Then we have:

$$b \geqslant d(x',D_\ell(f))^{\beta(\nu)}.$$

The polynomial β will be dependent on x, but upper bounds of the number of terms of β and of β' and upper bounds of the coefficients of β and β' will only be dependent on the constants and exponents in $L_1(f),\ldots,L_{p+1}(f)$, on upper bounds of the derivatives of f up to a certain order, on the number of terms of g and on upper bounds of the coefficients of g. Further all the coefficients of β and β' may be taken to be >1. At last, we can find an integer valued function $u(t)>1$, which only depends on bounds of the derivatives of f up to a certain order and on the constants and exponents of $L_1(f),\ldots,L_{p+1}(f)$, such that we have:

$$\deg g \leqslant \deg \beta \leqslant u(t)\deg g,$$
$$\deg g \leqslant \deg (\nu^m\beta(\nu)) \leqslant u(t)\deg g$$
$$\deg g \leqslant \deg \beta' \leqslant u(t)\deg g.$$

The neighbourhood U, where the above representative of f is defined, will also only be dependent on the coefficients and constants in $L_1(f),\ldots,L_{p+1}(f)$ and on bounds of the derivatives of f up to a certain order on some (larger) neighbourhood of 0.

Remark 5.2. The purpose of Lemma 5.1 is to show that we can approximate multiple points in the source by points satisfying inequality (2) of Proposition 4.2. (1) and (2) of 5.1 show that we can make the approximation as good as we want compared to the size of b (by making ν and t large).

Proof of Lemma 5.1. Recall the sets $V_p^\ell \subseteq V_{p-1}^\ell \subseteq\ldots\subseteq V_0^\ell \subseteq (R^p\times J^{p+1}(n,p))^\ell$ which is defined in the proof of Lemma 3.16. Choose $C_r = \frac{1}{2}$, and for some $\alpha_r>1$, $r = 0,\ldots,p$, define the neighbourhoods $N_r = N_{V_r^\ell}(C_r,\alpha_r,f)$.

(Recall the definition of such neighbourhoods in Lemma 3.8.) The α_r 's have to be chosen dependent of each other, but we will explain how to choose them later in this proof. Now, choose a representative of f defined on some neighbourhood U, such that the conclusion of Lemma 3.8 holds when we consider the sets V_r^ℓ, and our constants $C_r, \alpha_r, r=0,\ldots,p$. Suppose first that $x \in \bigcup_{r=0}^{p} N_r$. Repeating an argument in the proof of Lemma 3.16, we can find points $x_1,\ldots,x_q$, and integers $0 \leq r_1 < r_2 < \ldots < r_q \leq p$, which will satisfy the following:

5.1.1 $\qquad x_i \in V_{r_i}^\ell(f) = ((j^{p+1}f)^\ell)^{-1} V_{r_i}^\ell, \quad i = 1,\ldots,q.$

5.1.2 $\qquad \|x_{i-1} - x_i\| \leq C_{r_i} d(x_i, D_\ell(f))^{\alpha_{r_i}}, \quad i = 1,\ldots,q$

(here we define $x_0 = x$).

In addition, $x_q \notin N_r$ for any $r > r_q$. From 5.1.2 we get that

5.1.3 $$\|x - x_q\| \leq \tfrac{1}{2} \sum_{i=1}^{q} d(x_i, D_\ell(f))^{\alpha_{r_i}}.$$

Comparing $d(x_i, D_\ell(f))$ and $d(x_{i+1}, D_\ell(f))$, we will obtain that (provided U is sufficiently small and $\alpha_{r_{i+1}} > 1$) $d(x_i, D_\ell(f))^2 < d(x_{i+1}, D_\ell(f))$. (Arguments proving this can be found in the proof of 3.16.10.) So we have that $d(x_i, D_\ell(f))^{2\alpha_{r_{i+1}}} < d(x_{i+1}, D_\ell(f))^{\alpha_{r_{i+1}}}$. Let us suppose that we have chosen $2\alpha_{i+1} < \alpha_i$ for $i = 1,\ldots,p-1$. Since $r_{i+1} > r_i$, we get that $2\alpha_{r_{i+1}} < \alpha_{r_i}$. Let us suppose that U is so small that we have $d(x_i, D_\ell(f)) < 1$. Then we will get that

5.1.4 $$d(x_{i+1}, D_\ell(f))^{\alpha_{r_{i+1}}} > d(x_i, D_\ell(f))^{\alpha_{r_i}}.$$

5.1.3, 5.1.4, and the fact that $q \leq p+1$ implies that

5.1.5 $$\|x-x_q\| \leq \tfrac{1}{2}(p+1)d(x_q,D_\ell(f))^{\alpha_{r_q}}.$$

Let us also suppose that U is so small that we have $d(x_q,D_\ell(f))<2/(p+1)$. Then we get that

5.1.6 $$\|x-x_q\| \leq d(x_q,D_\ell(f))^{\alpha_{r_q}-1}.$$

Suppose first that $r_q<p$. Let $S = \{x_{q1},\dots,x_{q\ell}\}$ be the components of x_q. Since $x_q \in V_{r_q}(f)$, and $x_q \notin N_{r_q+1}$, we get that corank $B_f^\ell(S) = r_q$, and it follows from the conclusion of 3.8 that

5.1.7 $$b = b_f^{r_q}(S) \geq Kd(x_q,D_\ell(f))^{\beta}.$$

for suitable constants K, $\beta>0$. (See also the proof of 3.16.17 for a more complete version of this argument.) From the conclusion of Lemma 3.8 follows that we can choose $K = (1/\bar{K})(C_{r_q+1})^{\bar{\beta}}$, $\beta = P(\alpha_{r_q+1})$. Here $\bar{K}$ and $\bar{\beta}$ are some positive constants, and P is a positive polynomial. $\bar{\beta}$, $\bar{K}$, the degree, and the size of the coefficients of P are by 3.8 bounded by bounds only dependent of bounds of f's derivatives up to a certain order, and on the constants and exponents in $L_1(f),\dots,L_{p+1}(f)$, (and, of course, of the set $V^\ell_{r_q+1}$). Since the conclusion of 3.8 will be weakened by making $\bar{\beta}$, and P larger, we can find $\bar{\beta}$, and P working for every r. We can also suppose that all the non-zero coefficients of P are larger than 1.

If $r_q = p$, we have $x_q \in V_p^\ell(f)$. In this case it follows from $L_\ell(f)$ that

5.1.8 $$b > \tilde{K}d(x_q, D_\ell(f))^{\tilde{\beta}}$$

for some $\tilde{K}$, $\tilde{\beta}$ (only dependent on bounds of f's derivatives and on $L_\ell(f)$.) (A proof of this detail can be found in the proof of formula 3.16.26.)

Choosing $\bar{\beta}$ large if necessary, we may suppose that $(1/\bar{K})(\frac{1}{2})^{\bar{\beta}} < \tilde{K}$. If we also suppose that U is sufficiently small, we will get that $d(x_q, D_\ell(f)) < (1/\bar{K})(\frac{1}{2})^{\bar{\beta}}$. So we get that either

5.1.9 $$b > d(x_q, D_\ell(f))^{P(\alpha_{r_q+1})+1} \qquad (\text{if } r_q < p),$$

or that

5.1.10 $$b > d(x_q, D_\ell(f))^{\tilde{\beta}+1} \qquad (\text{if } r_q = p).$$

Now we will explain how to choose the α_i's. For any $\nu > 1$ choose $\alpha_p = \alpha_p(\nu) = \nu^{m_p}(g(\nu)+\tilde{\beta}+1)+1$ where $m_p = t \deg g$. Next, choose $\alpha_{p-1}(\nu) = \nu^{m_{p-1}}(P(\alpha_p(\nu))+1)+1$ where $m_{p-1} = t \deg(P(\alpha_p(\nu)))$, and inductively choose $\alpha_{i-1}(\nu) = \nu^{m_{i-1}}(P(\alpha_i(\nu))+1)+1$ where $m_{i-1} = t \deg(P(\alpha_i(\nu)))$. We can clearly suppose that $P(\nu) > 2\nu$. Since $\nu > 1$, it is then easy to see that this implies $\alpha_{i-1}(\nu) > 2\alpha_i(\nu)$. Now it follows from 5.1.6, and 5.1.9 or 5.1.10 that we can take $x' = x_q$, and $\beta(\nu) = P(\alpha_{r_q+1})+1$ if $r_q < p$, and $\beta(\nu) = g(\nu)+\tilde{\beta}+1 > \tilde{\beta}+1$ if $r_q = p$, to get (1) and (2) of this lemma to be satisfied.

The considerations above is done under the assumption $x \in \bigcup_{r=0}^{p} N_r$. Next we will determine $\beta'(\nu)$ such that

$$\sum_{i,j} \|f(x_i)-f(x_j)\|^2 + \sum_{i=1}^{\ell} J(f)(x_i) < d(x, D_\ell(f))^{\beta'(\nu)} \text{ implies } x \in \bigcup_{r=1}^{p} N_r.$$

Suppose $x \notin \bigcup_{r=0}^{p} N_r$, then $x \notin N_0$, and since we actually have

$V_0^\ell(f) = \sum(f)$, it follows from the conclusion of Lemma 3.8 that

5.1.11
$$\sum_{i,j} \|f(x_i)-f(x_j)\|^2 + \sum_i J(f)(x_i)$$
$$\geq (1/\bar{K})(\tfrac{1}{2})^{\bar{\beta}} d(x,D_\ell(f))^{P(\alpha_0)} \geq d(x,D_\ell(f))^{P(\alpha_0)+1}.$$

(Provided $d(x,D_\ell(f))<(1/\bar{K})(\tfrac{1}{2})^{\bar{\beta}}$.) From this it is clear that we can choose $\beta'(\nu) = P(\alpha_0(\nu))+1$.

From our construction of β and β', and the properties of the polynomial P of 3.8, follow that the number of terms of β and β' and the coefficients of β and β' have upper bounds only dependent on g, on bounds of the derivatives of f, and on the constants and exponents in $L_1(f),\dots,L_{p+1}(f)$. Since we have that $\deg \alpha_p = (t+1)\deg g$, $\deg \alpha_{i-1} = (t+1)\deg P(\alpha_i(\nu))$, and we have that $\beta = g+\tilde{\beta}+1$ or $\beta = P(\alpha_{r_q+1})+1$, and also that $\beta' = P(\alpha_0(\nu))+1$, we can find an integer valued function $u(t)>1$ which bounds the degree of β, $\nu^m\beta(\nu)$ and β' in the desired way. This u will only be dependent of the degree of P. It will therefore follow from the property of P, (recall Lemma 3.8) that u only is dependent of f, and $L_1(f),\dots,L_{p+1}(f)$ the way we claim in the text of this lemma.

At last it will follow from 3.8, and from the several inequalities we have put up in this proof, that the neighbourhood U also is only dependent of f and $L_1(f),\dots,L_{p+1}(f)$. This completes the proof of 5.1.

Our next lemma will show that we can approximate multiple points in the source with points satisfying the inequality (1) of 4.2 without destroying the approximation properties of 5.1.

Lemma 5.3. Suppose $f:(R^n,0) \to (R^p,0)$ satisfies $L_1(f),\dots,L_{p+1}(f)$. Let t be a positive integer, and let g be a positive polynomial in one variable. Then there exists a representative of f defined on a neighbourhood U such that the following holds:

Suppose $x = (x_1,\dots,x_\ell)\in(U)^\ell$ with $x_i\in\Sigma(f)$, $x_i\neq 0$, and $f(x_i) = f(x_j)$ for each i,j. Let $\nu>1$. Then there exist positive polynomials $\beta(\nu)$, $\tilde{\beta}(\nu)$, and there exists a point $x' = (x_1',\dots,x_\ell')\in(U)^\ell$ with $x_i'\in\Sigma(f)$ and $f(x_i') = f(x_j')$ for each i,j satisfying the following:

Let $k = \#\{x_i' \mid i = 1,\dots,\ell\}$, and let $\bar{x} = (\bar{x}_1,\dots,\bar{x}_k)\in(U)^k$ denote the point having the distinct components of x' as components (in some order). Then the following three conditions are satisfied:

(1) $\|x-x'\| \leq (p+1)d(\bar{x},D_k(f))^{\nu^m\beta(\nu)}$ where $m = t\deg\beta(\nu)$.

(2) Let $S = \{\bar{x}_1,\dots,\bar{x}_k\}$. let $r = \operatorname{corank} B_f^k(S)$ and let $b = b_f^r(S)$ be defined as in §2. Then we have

$$b \geq d(\bar{x},D_k(f))^{\beta(\nu)}.$$

(3) $d(\bar{x},D_k(f)) \geq (\min_{1\leq i\leq k}\|\bar{x}_i\|)^{\tilde{\beta}(\nu)}$.

The polynomials $\tilde{\beta}$, β will be dependent on x. However, upper bounds of the number of terms and of the coefficients can be put up independent of x. These bounds will be dependent of f and of g in the same way as the bounds of the β and β' of 5.1. Also all the coefficients in β and $\tilde{\beta}$ can be chosen >1. At last, there exists an integer d and an integer valued function $u(t)>1$, only dependent on bounds of the derivatives of f up to a certain order and on $L_1(f),\dots,L_{p+1}(f)$, such that we have:

$$\deg g \leq \deg \beta \leq u(t) \deg g,$$

$$\deg g \leq \deg(\nu^m \beta(\nu)) \leq u(t) \deg g,$$

$$\deg g \leq \deg \tilde{\beta} \leq d \deg \beta.$$

Also the size of the neighbourhood U will only be dependent on f and on $L_1(f),\ldots,L_{p+1}(f)$ in the same way as the U of 5.1.

Proof of Lemma 5.3. Let $g_s(\nu)$, $s = 1,\ldots,p-1$, be some positive polynomials in one variable, and let t_s, $s = 1,\ldots,p-1$, be some positive integers. g_s, t_s, $s = 1,\ldots,p$, have to be chosen dependent of each other, but we will explain how to choose them later in the proof. At the moment let us suppose that g_s and t_s already are chosen for $s = 1,\ldots,p$. Pick a representative of f defined in a neighbourhood U such that the conclusion of 5.1 holds with g_s and t_s in the hypothesis of 5.1 for $s = 1,\ldots,p-1$.

Since we have supposed that f satisfies $L_1(f),\ldots,L_{p+1}(f)$, we know that f is ∞-$\mathcal{K}$ determined (Lemma 3.4). We may therefore suppose that our representative of f (which we also denote f) is such that the conclusion of Sublemma 4.4 holds with this representative. Let $x = (x_1,\ldots,x_\ell) \in (U)^\ell$, $\ell > p$, be such that $f(x_i) = f(x_j)$ for all i,j, and $x_i \in \Sigma(f)$ for all i. Using the conclusion of 4.4, we get that there exists a $\lambda > 0$ (independent of x) such that

5.3.1 $$\|x_j\| \geq \|x_i\|^\lambda \quad \text{for} \quad 1 \leq i,j \leq \ell.$$

Now let $\nu_1,\ldots,\nu_p$ be p positive constants. $\nu_1,\ldots,\nu_p$ have to be chosen dependent of eachother. We will explain how to choose them later in the proof. At the moment let us suppose that $\nu_1,\ldots,\nu_p$ already are chosen.

Let $\nu>0$ be another constant, and let $\beta_\lambda(\nu) = \beta(\nu,\lambda)$ be the constant of the conclusion of Lemma 3.13 where $\lambda>0$ is the constant in 5.3.1. From Remark 3.15 follows that β_λ can be taken to be a polynomial in ν with coefficients, number of non-zero terms, and degree only dependent on λ, on bounds of the derivatives of f up to a certain order, and on the constants and exponents in $L_1(f),\dots,L_{p+1}(f)$. Since the λ also only is dependent on f in the way we have just described (this follows from the proof of 3.4 and 4.4), we can remove the dependence on λ in the polynomial $\beta_\lambda(\nu)$. Let us suppose that

5.3.2 $$d(x,D_\ell(f)) \leq \min_{1\leq i\leq \ell} \|x_i\|^{\beta_\lambda(\nu_1)}.$$

Now we will apply the conclusion of Lemma 3.13. (Recall that $x_i\in\int(f)$ so $J(f)(x_i) = 0$ for each i.) We can thus find a point $x^1 = (x_1^1,\dots,x_\ell^1)$ with $x_i^1\in\int(f)$, and $\ell_1<\ell$ distinct components such that if $\bar{x}^1 = (\bar{x}_1^1,\dots,\bar{x}_{\ell_1}^1)$ has these components as components (in some order), we have that

5.3.3 $$\|x-x^1\| \leq d(\bar{x}^1,D_{\ell_1}(f))^{\nu_1}.$$

Let us define $\tilde{x}^1 = (\tilde{x}_1^1,\dots,\tilde{x}_\ell^1)$ by putting $\tilde{x}^1 = x^1$. Now if

5.3.4 $$d(\bar{x}^1,D_{\ell_1}(f)) \leq (\min_{1\leq i\leq \ell_1} \|\bar{x}_i^1\|)^{\beta_\lambda(\nu_2)},$$

then we may apply Lemma 3.13 once again, and find points $x^2 = (x_1^2,\dots,x_{\ell_1}^2)$, and $\bar{x}^2 = (\bar{x}_1^2,\dots,\bar{x}_{\ell_2}^2)$ with $\ell_2<\ell_1$ satisfying the conclusion of 3.13.

Define $\tilde{x}^2 = (\tilde{x}_1^2,\dots,\tilde{x}_\ell^2)$ by $\tilde{x}_i^2 = x_j^2 \Leftrightarrow \tilde{x}_i^1 = \bar{x}_j^1$. Since $\tilde{x}^1$ and $\bar{x}^1$ have the same components, the definition of $\tilde{x}^2$ makes sense. If $d(\bar{x}^2,D_{\ell_2}(f) \leq \min_{1\leq i\leq \ell_2} \|\bar{x}_i^2\|^{\beta_\lambda(\nu_3)}$, then we repeat the arguments, and we find points x^3, $\bar{x}^3$, $\tilde{x}^3$, with ℓ_2, ℓ_3, and ℓ components respectively. We proceed this way until we obtain

$$5.3.5 \qquad d(\bar{x}^s, D_{\ell_s}(f)) \geq \left(\min_{1 \leq i \leq \ell_s} (\|\bar{x}^s_\ell\|)^{\beta_\lambda(\nu_{s+1})} \right) \quad \text{for an } s \leq p-1.$$

(From $L_1(f), \ldots, L_{p+1}(f)$ follows that f is stable outside 0, so we have that $p \geq \ell > \ell_1 > \ldots > \ell_s$, and we get that $s = p-1$ implies that $\ell_s = 1$. So if we suppose that $\beta_\lambda > 1$, then it is clear that 5.3.5 have to be satisfied for an $s < p-1$.)

To sum up, we have constructed points $x^1, \ldots, x^s$, and $\bar{x}^1, \ldots, \bar{x}^s$ such that we have

$$5.3.6 \qquad \|\bar{x}^{i-1} - x^i\| \leq d(\bar{x}^i, D_{\ell_i}(f))^{\nu_i}, \quad i = 1, \ldots, s,$$

and such that 5.3.5 holds. (In 5.3.6 we define $\bar{x}^0 = x$.) We have also defined the points $\tilde{x}^1, \ldots, \tilde{x}^s \in (R^n)^\ell$ by $\tilde{x}^i_i = x^i_j \Leftrightarrow \tilde{x}^{i-1}_i = \bar{x}^{i-1}_j$ for $i = 2, \ldots, s$, and we have that $\tilde{x}^1 = x^1$. Now we want to estimate $\|x - \tilde{x}^s\|$. To do so, we will first compare $d(\bar{x}^i, D_{\ell_i}(f))$ and $d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))$ for $i < s$. Let $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_{\ell_{i+1}})$ be a point in $D_{\ell_{i+1}}(f)$ such that $\|\bar{x}^{i+1} - \bar{z}\| = d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))$. Define $z = (z_1, \ldots, z_{\ell_i})$ by $z_i = \bar{z}_j \Leftrightarrow x^{i+1}_i = \bar{x}^{i+1}_j$. (Recall that x^{i+1}, $\bar{x}^{i+1}$ have the same components.) Since $\ell_i > \ell_{i+1}$, we get that $z \in D_{\ell_i}(f)$.

Now we have from 5.3.6 that

$$5.3.7 \qquad d(\bar{x}^i, D_{\ell_i}(f)) \leq \|\bar{x}^i - z\| \leq \|\bar{x}^i - x^{i+1}\| + \|x^{i+1} - z\|$$

$$\leq d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))^{\nu_{i+1}} + \|x^{i+1} - z\|.$$

Since $\|\bar{x}^{i+1}-\bar{z}\| = d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))$, and $\bar{x}^{i+1}-\bar{z}$ and $x^{i+1}-z$ have the same components, we get that

$\|x^{i+1}-z\| \leq \ell_i d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f)) < pd(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))$. From above, and from 5.3.7 we get that

$d(\bar{x}^i, D_{\ell_i}(f)) \leq d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))^{\nu_{i+1}} + pd(\bar{x}^{i+1}, D_{\ell_{i+1}}(f))$. Supposing $d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f)) < 1$, $\nu_{i+1} > 1$, and $d(\bar{x}^i, D_{\ell_i}(f)) < 1/(p+1)$, we deduce that

5.3.8 $$d(\bar{x}^i, D_{\ell_i}(f))^2 \leq d(\bar{x}^{i+1}, D_{\ell_{i+1}}(f)).$$

Now we have by definition, that $\tilde{x}^{i-1}-\tilde{x}^i$ have the same components as $\bar{x}^{i-1}-x^i$, and it follows from 5.3.6 that

$\|\tilde{x}^{i-1}-\tilde{x}^i\| \leq \ell d(\bar{x}^i, D_{\ell_i}(f))^{\nu_i}$. So by the triangle inequality, we get that $\|x-\tilde{x}^s\| \leq \sum_{i=1}^{s} \ell d(\bar{x}^i, D_{\ell_i}(f))^{\nu_i}$. Now suppose that $\nu_{i-1} > 2\nu_i$ for $i = 2,\ldots,p$. Then it follows from 5.3.8 that

5.3.9 $$\|x-\tilde{x}^s\| \leq \ell s d(\bar{x}^s, D_{\ell_s}(f))^{\nu_s} \leq p^2 d(\bar{x}^s, D_{\ell_s}(f))^{\nu_s}.$$

Now we want to apply Lemma 5.1 with the point $\bar{x}^s$, and with g_s, t_s in the hypothesis. From Lemma 3.13 follows that each component of $\bar{x}^s$ is singular, we therefore only have to estimate $\sum_{1 \leq i,j \leq \ell_s} \|f(\bar{x}_i^s)-f(\bar{x}_j^s)\|^2$ to see if the hypothesis of 5.1 is satisfied. Since all the components in $\bar{x}^s$ appear as components in $\tilde{x}^s$, we can estimate $\sum_{1 \leq i,j \leq \ell} \|f(\tilde{x}_i^s)-f(\tilde{x}_j^s)\|^2$ instead. Let $K>0$ be a constant bounding the derivatives of f. Then we get from 5.3.9 using the triangle inequality that

5.3.10 $$\|f(\tilde{x}_i^s)-f(\tilde{x}_j^s)\| \leq \|f(\tilde{x}_i^s)-f(x_i)\| + \|f(x_i)-f(x_j)\| + \|f(x_j)-f(\tilde{x}_j^s)\| \leq 2p^2Kd(\bar{x}^s,D_{\ell_s}(f))^{\nu_s}.$$

(In the estimate above recall that $f(x_i) = f(x_j)$ from the hypothesis of this Lemma.) 5.3.10 implies that

5.3.11 $$\sum_{1\leq i,j\leq \ell_s} \|f(\bar{x}_i^s)-f(\bar{x}_j^s)\|^2 \leq 4p^6K^2d(\bar{x}^s,D_{\ell_s}(f))^{2\nu_s} \leq d(\bar{x}^s,D_{\ell_s}(f))^{2\nu_s-1}.$$

(Provided that $d(\bar{x}^s,D_{\ell_s}(f))<1/(4p^6K^2)$.)

Now, let $\beta_s'(\nu)$ be the polynomial of Lemma 5.1 we get from using g_s and t_s in the hypothesis of 5.1. Suppose that $2\nu_s \geq \beta_s'(\nu)+1$. Then it follows from 5.3.11 that the hypothesis of 5.1 is satisfied. We can then find a polynomial $\beta_s(\nu)$, and a point $\bar{x}' = (\bar{x}_1',\ldots,\bar{x}_{\ell_s}')$ such that

5.3.12 $$\|\bar{x}^s-\bar{x}'\| \leq d(\bar{x}',D_{\ell_s}(f))^{\nu^{m_s}\beta_s(\nu)} \quad \text{where } m_s=t_s\deg\beta_s.$$

Also x' and β_s satisfy the remaining part of the conclusion of 5.1. Now using 5.3.12, and the same sort of arguments as those we used when we deduced 5.3.8, we deduce that if U is small, then

5.3.13 $$d(\bar{x}^s,D_{\ell_s}(f))^2 \leq d(\bar{x}',D_{\ell_s}(f)).$$

In the same manner we can also deduce that

5.3.14 $$\min_{1\leq i\leq \ell_s} \|\bar{x}_i'\|^2 \leq \min_{1\leq i\leq \ell_s} \|\bar{x}_i^s\|.$$

Then 5.3.5, 5.3.13 and 5.3.14 implies that

5.3.15 $$d(\bar{x}',D_{\ell_s}(f)) \geq (\min_{1\leq i\leq \ell_s} \|\bar{x}_i'\|)^{4\beta_\lambda(\nu_{s+1})}.$$

Now we want to determine $\nu_1,\ldots,\nu_p$, $g_s(\nu)$ and t_s. Let us start by choosing $\nu_p = \nu>1$, and $g_{p-1}(\nu) = g(\nu)$, $t_{p-1} = t$. Define $\tilde{\beta}_{p-1}(\nu) = 4\beta_\lambda(\nu)$. Let us suppose that $\deg \beta_\lambda = d$. So we have $\deg \tilde{\beta}_{p-1} = d$. We get induced the polynomial β'_{p-1} using t_{p-1}, g_{p-1} in the hypothesis of Lemma 5.1.

Next put $\nu_{p-1} = \beta'_{p-1}(\nu)+1$, and put $g_{p-2}(\nu) = \nu_{p-1}$, $t_{p-2} = t$. Put $\tilde{\beta}_{p-2}(\nu) = 4\beta_\lambda(\nu_{p-1})$. We get induced $\beta'_{p-2}(\nu)$ using g_{p-2}, t_{p-2} in the hypothesis of 5.1. Next define $\nu_{p-2} = \beta'_{p-2}(\nu)+1$ and $g_{p-3}(\nu) = \nu_{p-2}$ and so on. Having defined $g_i(\nu)$, $t_i = t$ we get induced $\beta_i'(\nu)$, and we define $\nu_i = \beta_i'(\nu)+1$, $g_{i-1}(\nu) = \nu_i$, and $t_{i-1} = t$. We also put $\tilde{\beta}_{i-1}(\nu) = 4\beta_\lambda(\nu_i)$. Since we have $2\nu_s>\nu_s= \beta_s'(\nu)+1$, it follows from 5.3.11 that the hypothesis of 5.1 will be satisfied. Modifying the proof of 5.1 if necessary, we can always obtain $\beta'(\nu)>2g(\nu)$ in Lemma 5.1. Then we also get that $\nu_{i-1} = \beta'_{i-1}(\nu)+1>2g_{i-1}(\nu) = 2\nu_i$. With such a chose of $\nu_1,\ldots,\nu_p$, $g_1,\ldots,g_{p-1}$, $t_1,\ldots,t_{p-1}$ it follows that the inequalities we have deduced so far are valid.

Now define $x' = (x_1',\ldots,x_\ell')$ by $x_i' = \bar{x}_j' \Leftrightarrow \tilde{x}_i^s = \bar{x}_j^s$. Then we have, by the triangle inequality using 5.3.9 and 5.3.12, that

5.3.16 $$\|x-x'\| \leq p^2 d(\bar{x}^s,D_{\ell_s}(f))^{\nu_s} + \ell d(\bar{x}',D_{\ell_s}(f))^{\nu^{m_s}\beta_s(\nu)}.$$

(Note that $\tilde{x}^s-x'$ has, by construction, the same components as $\bar{x}^s-\bar{x}'$, so we can use 5.3.12 to estimate $\|\tilde{x}^s-x'\|$.)

Now, we have $\nu_s = \beta_s'(\nu)+1$. From the proof of 5.1 it is easy to see that $\beta_s'(\nu)>2\nu^{m_s}\beta_s(\nu)$. (To see this recall the definition of $\beta'(\nu)$ in 5.1, and use that we have supposed $P(\nu)>2\nu$, $\nu>1$ in the proof of 5.1.) From 5.3.13 now follows that

5.3.17 $$d(\bar{x}^s, D_{\ell_s}(f))^{\nu^{s-1}} < d(\bar{x}', D_{\ell_s}(f))^{\nu^{m_s}\beta_s(\nu)}.$$

Let us suppose that $d(\bar{x}^s, D_{\ell_s}(f)) < 1/p^2$. Since $\ell < p$, 5.3.16 and 5.3.17 implies that

5.3.18 $$\|x-x'\| \leq (p+1)d(\bar{x}', D_{\ell_s}(f))^{\nu^{m_s}\beta_s(\nu)}.$$

Put $\beta(\nu) = \beta_s(\nu)$, $\tilde{\beta}(\nu) = \tilde{\beta}_s(\nu)$, and let $\bar{x} = \bar{x}^s$. Now condition (1) of this Lemma follows from 5.3.18, condition (2) follows from the conclusion of Lemma 5.1, and condition (3) follows from 5.3.15 and the definition of $\tilde{\beta}(\nu)$.

The claims on the degree of $\beta(\nu)$ and $\nu^m\beta(\nu)$ follow from the corresponding claims of Lemma 5.1 and our construction of $g_i(\nu)$, $i = p-1,\dots,1$. Since we have from the definition of $\tilde{\beta}_s$ that $\deg \tilde{\beta}_s = d \deg g_s$ where $d = \deg \beta_\lambda$, and it follows from 5.1 that $\deg g_s \leq \deg \beta_s$, we get that $\deg \tilde{\beta}_s \leq d \deg \beta_s$. Also the claims about the number of terms, and estimates on the coefficients, of β, $\tilde{\beta}$ follows from corresponding properties of the polynomials in 5.1, already mentioned properties of the polynomial β_λ, and the inductive way we construct g_p, g_{p-1} etc. It also follows from 5.1, from several inequalities we have put up in this proof, and from Remark 3.15, that U is only dependent of bounds of the derivatives of f up to a certain order, and of $L_1(f),\dots,L_{p+1}(f)$.

Lemma 5.3 show us that it is possible to approximate points in the source with other points such that inequalities of the type (1) and (2) of Proposition 4.2 hold for these points at the same time. However, the way Proposition 4.2 is formulated, we need to approximate in the target. Suppose that $y \in \Sigma(f)$, $f^{-1}(y) \cap \Sigma(f) = \{x_1, \ldots, x_\ell\}$. Then one can use Lemma 5.3, and approximate $x = (x_1, \ldots, x_\ell)$ with a point $x' = (x_1', \ldots, x_\ell')$ with $k \leq \ell$ distinct components such that the corresponding $\bar{x}' = (\bar{x}_1, \ldots, \bar{x}_k)$ satisfies inequality (1) and (2) of 4.2. (By (2) and (3) of 5.3.) Putting $\bar{y} = f(\bar{x}_1) = \ldots = f(\bar{x}_k)$, then clearly $\|y-\bar{y}\|$ satisfies an inequality of type (1) of 5.3. However, it might happen that $f^{-1}(\bar{y}) \cap \Sigma(f) = (\bar{x}_1, \ldots, \bar{x}_k, \bar{x}_{k+1}, \ldots, \bar{x}_q)$ with $q>k$. So, if we define $b = b_f^r(\{\bar{x}_1, \ldots, \bar{x}_q\})$, and $d((\bar{x}_1, \ldots, \bar{x}_q), D_q(f))$ and formulate (1), (2), (3) of 5.3 in terms of these numbers, then some of the inequalities (1), (2) or (3) may be violated. Since 4.2 actually requires approximation in the target, we need to have a version of 5.3 which give us that (1), (2) and (3) still hold, when we formulate (1), (2) and (3) with respect to $b = b_f^r(\{\bar{x}_1, \ldots, \bar{x}_q\})$, and $d((\bar{x}_1, \ldots, \bar{x}_q), D_q(f))$, where $q = f^{-1}(\bar{y}) \cap \Sigma(f)$. Such a lemma will be the following:

Lemma 5.4. Let $f:(R^n,0) \to (R^p,0)$ satisfy $L_1(f), \ldots, L_{p+1}(f)$. Let t be a positive integer, and let g be a positive polynomial in one variable. Then there exist a representative of f defined on a neighbourhood U and positive constants M, K such that the following holds:

Let $y \in f(\Sigma(f)) - \{0\}$, $f^{-1}(y) \cap \Sigma(f) = \{x_1, \ldots, x_\ell\}$ and $x = (x_1, \ldots, x_\ell) \in (R^n)^\ell$. Let $\nu > M$. Then there exist a point $y' \in f(\Sigma(f)) - \{0\}$ and positive polynomials $\beta(\nu)$, $\tilde{\beta}(\nu)$ satisfying the following:

(1) Let $S' = \{x_1', \ldots, x_k'\} = f^{-1}(y') \cap \Sigma(f)$, and let $x' = (x_1', \ldots, x_k') \in (R^n)^k$. Then we have

$$d(x',D_k(f)) \geq \min_{1\leq i\leq k} \|x_i'\|^{\tilde{\beta}(\nu)}.$$

(2) Let $B = B_f^k(S')$ be the linear map of §2, and suppose that $r = \operatorname{corank} B$. Let $b = b_f^r(S')$. Then we have

$$b \geq d(x',D_k(f))^{\beta(\nu)}.$$

(3) There exists an integer m such that $m\geq t \deg \beta$, and a point $\hat{x} = (\hat{x}_1,\ldots,\hat{x}_\ell)$ such that $\hat{x}_i \in S'$, $i = 1,\ldots,\ell$, and such that the following two inequalities hold:

$$\|y-y'\| \leq Kd(x',D_k(f))^{\nu^m\beta(\nu)}$$

$$\|x-\hat{x}\| \leq Kd(x',D_k(f))^{\nu^m\beta(\nu)}.$$

U, K are only dependent on bounds of the derivatives of f and of the constants and exponents in $L_1(f),\ldots,L_{p+1}(f)$. M, upper bounds of the coefficients of β and $\tilde{\beta}$ and upper bounds of the number of terms of β and $\tilde{\beta}$ are dependent on upper bounds of the coefficients of g, of the number of terms in g, of bounds of the derivatives of f up to a certain order and of the constants and exponents in $L_1(f),\ldots,L_{p+1}(f)$. All the non-zero coefficients in β and $\tilde{\beta}$ can be taken to be ≥ 1. Also there exist an integer d and an integer valued function $u(t)>1$ such that we have: $\deg g\leq \deg \tilde{\beta} \leq d \deg \beta$, $\deg g \leq \deg \beta \leq u(t) \deg g$, $\deg g \leq \deg(\nu^m\beta(\nu)) \leq u(t) \deg g$. u and d are only dependent on bounds of the derivatives of f up to a certain order and of the constants and exponents in $L_1(f),\ldots,L_{p+1}(f)$.

<u>Proof of Lemma 5.4</u>. We start by choosing positive integers t_s and polynomials g_s, $s = 1,\dots,p$. We will use t_s and g_s in the hypothesis of Lemma 5.3. The way our proof works, the t_s and g_s, $s = 1,\dots,p$, have to be chosen dependent of each other in a rather complicated way. We will explain how this choise has to be made later in the proof. So let us at the moment only suppose that the t_s and g_s are chosen for each $s = 1,\dots,p$.

Choose a representative $f:(U,0) \to (R^p,0)$ for f such that Lemma 5.3 holds for this f, for every t_s and g_s in the hypothesis of 5.3. Also let $\nu > 1$ be some constant. Let $y \in f(\Sigma(f))$, and define $x = (x_1,\dots,x_\ell)$ where $\{x_1,\dots,x_\ell\} = f^{-1}(y) \cap \Sigma(f)$. Let us first use Lemma 5.3 with t_1, g_1 in the hypothesis to approximate x by a point $x^1 = (x_1^1,\dots,x_\ell^1)$ having $k_1 \leq \ell$ distinct components. Let $\bar{x}^1 = (x_1^1,\dots,\bar{x}_{k_1}^1)$ have these components as components. Now x^1, $\bar{x}^1$ satisfy the conclusion of 5.3 for some polynomials $\beta_1(\nu)$, $\tilde{\beta}_1(\nu)$ and an integer m_1, such that $m_1 = t_1 \deg \beta_1$. Put $y^1 = f(x_1^1) = \dots = f(x_\ell^1)$. Also define $\hat{x}^1 = (\hat{x}_1^1,\dots,\hat{x}_\ell^1)$ by $\hat{x}^1 = x^1$. Put $\ell_1 = \# f^{-1}(y^1) \cap \Sigma(f)$. Then it might happen that $\ell_1 > k_1$. If so is the case, let $f^{-1}(y^1) \cap \Sigma(f) = \{\tilde{x}_1^1,\dots,\tilde{x}_{\ell_1}^1\}$ where $\tilde{x}_i^1 = \bar{x}_i^1$ for $i \leq k_1$. Put $\tilde{x}^1 = (\tilde{x}_1^1,\dots,\tilde{x}_{\ell_1}^1) \in (U)^{\ell_1}$. Now we want to apply Lemma 5.3 again, and this time we approximate $\tilde{x}^1$ with a point $x^2 = (x_1^2,\dots,x_{\ell_1}^2)$ with k_2 distinct components, using t_2, g_2 in the hypothesis of Lemma 5.3. We want to find conditions that will give us $k_2 > k_1$.

Let $\bar{x}^2 = (\bar{x}_1^2,\dots,\bar{x}_{k_2}^2)$ have the distinct components of x^2 as components. Let $\beta_2(\nu)$, $m_2 = t_2 \deg \beta_2$ be the polynomial and integer in the conclusion of 5.3. Then we have from the conclusion of 5.3 that

5.4.1 $$\|x^2 - \tilde{x}^1\| \leq (p+1) d(\bar{x}^2, D_{k_2}(f))^{\nu^{m_2}\beta_2(\nu)}.$$

Let us compare $d(\bar{x}^1, D_{k_1}(f))$ and $d(\bar{x}^2, D_{k_2}(f))$. By the conclusion of 5.3, we have that

5.4.2 $$d(\bar{x}^1, D_{k_1}(f)) \geq (\min_{1 \leq i \leq k_1} \|\bar{x}_i^1\|)^{\tilde{\beta}_1(\nu)}.$$

Now, for each $i = 1, \ldots, \ell_1$, we have that

5.4.3 $$\|x_i^2\| \leq \|\tilde{x}_i^1\| + \|\tilde{x}_i^1 - x_i^2\|$$

$$\leq \|\tilde{x}_i^1\| + (p+1) d(\bar{x}^2, D_{k_2}(f))^{\nu^{m_2 \beta_2(\nu)}}.$$

From the proof of Lemma 5.3, it is not hard to see that we can arrange ourselves such that $\nu^{m_2 \beta_2(\nu)} \geq 3$. Let us also suppose that $d(\bar{x}^2, D_{k_2}(f)) < 1/(p+1)$. Then we get that

5.4.4 $$(p+1) d(\bar{x}^2, D_{k_2}(f))^{\nu^{m_2 \beta_2(\nu)}} < d(\bar{x}^2, D_{k_2}(f))^2 < \tfrac{1}{2} \|x_i^2\|.$$

Let us also suppose that $\|x_i^2\| < \frac{1}{2}$. Then from 5.4.3, and 5.4.4 it follows that

5.4.5 $$\|x_i^2\|^2 \leq \|\tilde{x}_i^1\| \quad \text{for} \quad i = 1, \ldots, \ell_1.$$

From 5.4.2 and 5.4.5 it follows that

5.4.6 $$d(\bar{x}^2, D_{k_2}(f))^2 \leq \min_{1 \leq i \leq \ell_1} \|x_i^2\|^2 \leq \min_{1 \leq i \leq \ell_1} \|\tilde{x}_i^1\|$$

$$\leq d(\bar{x}^1, D_{k_1}(f))^{1/\tilde{\beta}_1(\nu)}.$$

We thus get that

5.4.7 $$d(\bar{x}^2, D_{k_2}(f))^{2\tilde{\beta}_1(\nu)} \leq d(\bar{x}^1, D_{k_1}(f)).$$

Now, let $S_1 = \{\bar{x}_1^1, \ldots, \bar{x}_{k_1}^1\}$, let $r_1 = \operatorname{corank} B_f^{k_1}(S_1)$, and put $b_1 = b_f^{r_1}(S_1)$. Recall Lemma 2.3. In that lemma we proved that there exists a constant C_f (only dependent on bounds of f's $p+2$ order derivatives) such that if we consider small balls U_i

at $\bar{x}_i^1$, $i = 1,\ldots,k_1$, with radius $(1/C_f)b_1^{\frac{1}{2}}$, then these balls are disjoint. Actually, from the proof of 2.3 follows that $f^{-1}(y^1)\cap U_i \cap \Sigma(f) = \{\bar{x}_i^1\}$. From the conclusion of 5.3 we had that

5.4.8 $$b_1 \geq d(\bar{x}^1, D_{k_1}(f))^{\beta_1(\nu)}.$$

Let us suppose that $d(\bar{x}^1, D_{k_1}(f)) < 1/C_f$. It will now follow from the above mentioned properties of U_i, and the fact that $f(\tilde{x}_i^1) = f(\tilde{x}_j^1)$ for all i,j, that if $i \leq k_1$ and $j = 1,\ldots,\ell_1$, $i \neq j$, then we have

5.4.9 $$\|\tilde{x}_i^1 - \tilde{x}_j^1\| \geq d(\bar{x}^1, D_{k_1}(f))^{(\beta_1(\nu)/2)+1}.$$

Let us suppose that it is possible to choose t_i, g_i in such a way that we always obtain

5.4.10 $$\nu^{m_2}\beta_2(\nu) - 1 > 2\tilde{\beta}_1(\nu)\beta_1(\nu).$$

Then it follows from 5.4.7 and 5.4.10 that

5.4.11 $$d(\bar{x}^2, D_{k_2}(f))^{\nu^{m_2}\beta_2(\nu)-1} < d(\bar{x}^1, D_{k_1}(f))^{\beta_1(\nu)}.$$

(Provided $d(\bar{x}^2, D_{k_2}(f)), d(\bar{x}^1, D_{k_1}(f)) < 1$.)
From 5.4.1 and 5.4.11 it follows that for each i, x_i^2 is contained in the ball at $\tilde{x}_i^1$ with radius $d(\bar{x}^1, D_{k_1}(f))^{\beta_1(\nu)}$. (Provided $d(\bar{x}_2, D_{k_2}(f)) < 1/(p+1)$.) If we consider the k_1+1 points $\tilde{x}_1^1,\ldots,\tilde{x}_{k_1}^1$, $\tilde{x}_j^1$ where j is some index $j > k_1$, it follows from 5.4.9 that these balls are disjoint provided $d(\bar{x}^1, D_{k_1}(f))^{\beta_1(\nu)} < \frac{1}{2} d(\bar{x}^1, D_{k_1}(f))^{(\beta_1(\nu)/2)+1}$. If we suppose that $d(\bar{x}^1, D_{k_1}(f)) < \frac{1}{2}$, then a sufficient condition for this is that $\beta_1(\nu) > (\beta_1(\nu)/2)+2$. It is easy to see from the proofs of 5.1 and of 5.3, that we can arrange ourselves such that we always have $\beta_1(\nu) > 4$. This will imply the above inequality. Supposing 5.4.10, we then get that the balls at $\tilde{x}_1^1,\ldots,\tilde{x}_{k_1}^1$, $\tilde{x}_j^1$, $j > k_1$ with radius

$d(\bar{x}^1, D_{k_1}(f))^{\beta_1(\nu)}$ are all disjoint. Since each $x_1^2, \ldots, x_{k_1}^2$, x_j^2 belongs to one such ball, none of the $x_1^2, \ldots, x_{k_1}^2$, x_j^2 can coincide. It follows that $x^2 = (x_1^2, \ldots, x_{\ell_1}^2)$ has $k_2 > k_1$ distinct components. So, from the arguments above, we get that 5.4.10 is a sufficient condition for obtaining $k_2 > k_1$.

Now put $y^2 = f(\bar{x}_1^2) = \ldots = f(\bar{x}_{k_2}^2)$, and define $\hat{x}^2 = (\hat{x}_1^2, \ldots, \hat{x}_\ell^2)$ by $\hat{x}_i^2 = x_j^2 \Leftrightarrow \hat{x}_i^1 = \tilde{x}_j^1$. (Recall that the components of $\hat{x}^1$ are those of $\bar{x}^1$ hence occur among the components of $\tilde{x}^1$.)

Suppose $\# f^{-1}(y^2) \cap \Sigma(f) = \ell_2 > k_2$. Put $f^{-1}(y^2) \cap \Sigma(f) = \{\tilde{x}_1^2, \ldots, \tilde{x}_{\ell_2}^2\}$ where $\tilde{x}_i^2 = \bar{x}_i^2$ for $i \leq k_2$, and let $\tilde{x}^2 = (\tilde{x}_1^2, \ldots, \tilde{x}_{\ell_2}^2)$. Again use Lemma 5.3 with t_3, g_3 in the hypothesis to construct the point $x^3 = (x_1^3, \ldots, x_{\ell_2}^3)$ with k_3 distinct components, and the point $\bar{x}^3 = (\bar{x}_1^3, \ldots, \bar{x}_{k_3}^3)$ such that the conclusion of 5.3 holds this time with polynomials $\beta_3(\nu)$, $\tilde{\beta}_3(\nu)$, and $m_3 = t_3 \deg \beta_3(\nu)$. Again, if an inequality of the type 5.4.10 (but this time involving m_3, $\beta_3(\nu)$, $\tilde{\beta}_2(\nu)$, $\beta_2(\nu)$) holds, then we have $k_3 > k_2$. Define $y^3 = f(\bar{x}_1^3) = \ldots = f(\bar{x}_{k_3}^3)$, and $\hat{x}^3 = (\hat{x}_1^3, \ldots, \hat{x}_\ell^3)$ by $\hat{x}_i^3 = x_j^3 \Leftrightarrow \hat{x}_i^2 = \tilde{x}_j^2$.

Proceeding this way, we can for each q construct points $x^q = (x_1^q, \ldots, x_{\ell_{q-1}}^q)$, and $\bar{x}^q = (\bar{x}_1^q, \ldots, \bar{x}_{k_q}^q)$ which approximate a point $\tilde{x}^{q-1} = (\tilde{x}_1^{q-1}, \ldots, \tilde{x}_{\ell_{q-1}}^{q-1})$ such that the conclusion of 5.3 holds. For each index q we use t_q, g_q in the hypothesis of 5.3, and the conclusion of 5.3 holds in terms of polynomials β_q, $\tilde{\beta}_q$, and $m_q = t_q \deg \beta_q$. Put $y^q = f(\bar{x}_1^q) = \ldots = f(\bar{x}_{k_q}^q)$, and define $\hat{x}^q = (\hat{x}_1^q, \ldots, \hat{x}_\ell^q)$ by $\hat{x}_i^q = x_j^q \Leftrightarrow \hat{x}_i^{q-1} = \tilde{x}_j^{q-1}$ (where $\tilde{x}^{q-1}$ is defined in the previous step). Suppose that we at each step has satisfied the inequality

$$5.4.12 \qquad \nu^{m_q \beta_q(\nu) - 1} > 2\tilde{\beta}_{q-1}(\nu)\beta_{q-1}(\nu).$$

Then we get that $k_q > k_{q-1}$, reasoning exactly as in the case $q = 2$. Since f satisfies $L_1(f),\ldots,L_{p+1}(f)$ and therefore is stable outside 0, and we have that each $\bar{x}_1^q,\ldots,\bar{x}_{k_q}^q$ are different from 0 (otherwize $d(\bar{x}^q, D_{k_q}(f)) = 0$, and the approximation is impossible), it is clear that $k_q \leq p$.

Since $k_1 < k_2 < \ldots < k_q$, we must obtain $\# f^{-1}(y^s) \cap \Sigma(f) = k_s$ for an $s \leq p$. It follows that our repeated use of Lemma 5.3 can be terminated after $s \leq p$ steps. So for $q = 1,\ldots,s$, $s \leq p$, we have constructed points $x^q = (x_1^q,\ldots,x_{\ell_{q-1}}^q)$, $\bar{x}^q = (\bar{x}_1^q,\ldots,\bar{x}_{k_q}^q)$, $\hat{x}^q = (\hat{x}_1^q,\ldots,\hat{x}_\ell^q)$, and also $\tilde{x}^q = (\tilde{x}_1^q,\ldots,\tilde{x}_{\ell_q}^q)$ for $q<s$, such that the following three conditions hold:

5.4.13 $$\|\tilde{x}^{q-1} - x^q\| \leq (p+1)\, d(\bar{x}^q, D_{k_q}(f))^{\nu^{m_q}\beta_q(\nu)}$$

where $m_q = t_q \deg \beta_q$.

5.4.14 If $S_q = \{\bar{x}_1^q,\ldots,\bar{x}_{k_q}^q\}$, $r_q = \operatorname{corank} B_f^{k_q}(S_q)$, and

$$b_q = b_f^{r_q}(S_q), \text{ then we have that } b_q \geq d(\bar{x}^q, D_{k_q}(f))^{\beta_q(\nu)}.$$

5.4.15 $$d(\bar{x}^q, D_{k_q}(f)) \geq \left(\min_{1 \leq i \leq k_q} \|x_i^q\|\right)^{\tilde{\beta}_q(\nu)}.$$

Here β_q, $\tilde{\beta}_q$ are the polynomials we get from applying 5.3 with t_q, g_q in the hypothesis.

Next we want to estimate $\|y - y^s\|$, $\|x - \hat{x}^s\|$ where y^s is the image of $\bar{x}_1^s,\ldots,\bar{x}_{k_s}^s$. Since we may suppose that the derivatives of f are bounded by some constant, it follows from 5.4.13 that for a $K>0$ we have that

5.4.16 $$\|y^{q-1}-y^q\| \le Kd(\bar{x}^q, D_{k_q}(f))^{\nu^{m_q}\beta_q(\nu)}, \quad q = 1,\ldots,s.$$

(For $q = 1$, define $y^0 = y$.)

Let us compare $d(\bar{x}^{q-1}, D_{k_{q-1}}(f))$, and $d(\bar{x}^q, D_{k_q}(f))$ for $q = 2,\ldots,s$ this time in the other direction. Since $\tilde{x}_i^{q-1} = \bar{x}_i^{q-1}$ for $i<k_{q-1}$, 5.4.13 implies that

$\|\bar{x}_i^{q-1}-x_i^q\| \le (p+1)d(\bar{x}^q, D_{k_q}(f))^{\nu^{m_q}\beta_q(\nu)}$ for $i = 1,\ldots,k_{q-1}$. From the above inequality we easily deduce that

5.4.17 $$\|\bar{x}_i^{q-1}\|^2 \le \|x_i^q\| \quad \text{for } i = 1,\ldots,k_{q-1},$$

provided $d(\bar{x}^q, D_{k_q}(f))<1/(p+1)$, $\|\bar{x}_i^{q-1}\|<\frac{1}{2}$ and $\nu^{m_q}\beta_q(\nu)\ge 3$. Since f satisfies $L_1(f),\ldots,L_{p+1}(f)$ (and therefore is ∞-$\mathcal{K}$ determined by 3.4), we can use Sublemma 4.4, and deduce

5.4.18 $$\|f(x)\| \ge \|x\|^\lambda \quad \text{for } x\in\Sigma(f),$$

5.4.19 $$\|f(x)\|^2 \le \|x\| \quad \text{for all } x,$$

5.4.20 $$\|x_j^q\| \ge \|x_i^q\|^\lambda \quad \text{for each } 1\le i,j\le \ell_{q-1}$$

for a suitble constant λ.

Now it follows from 5.4.15, 5.4.17 and 5.4.20 that we have

5.4.21 $$d(\bar{x}^{q-1}, D_{k_{q-1}}(f)) \le \min_{1\le i\le k_{q-1}} \|\bar{x}_i^{q-1}\| \le \min_{1\le i\le k_{q-1}} \|x_i^q\|^{\frac{1}{2}}$$

$$\le \min_{1\le i\le k_q} \|\bar{x}_i^q\|^{1/(2\lambda)} \le d(\bar{x}^q, D_{k_q}(f))^{1/(2\lambda\tilde{\beta}_q(\nu))}.$$

Let us suppose that it is possible to choose t_q, g_q, $q = 1,\ldots,p$ such that we always get that

5.4.22 $$\nu^{m_{q-1}}\beta_{q-1}(\nu) \geq 2\lambda\nu^{m_q}\beta_q(\nu)\tilde{\beta}_q(\nu).$$

Then 5.4.21 implies that

$d(\bar{x}^{q-1},D_{k_{q-1}}(f))^{\nu^{m_{q-1}}\beta_{q-1}(\nu)} \leq d(\bar{x}^q,D_{k_q}(f))^{\nu^{m_q}\beta_q(\nu)}$. (Provided $d(\bar{x}^{q-1},D_{k_{q-1}}(f))<1$ and $d(\bar{x}^q,D_{k_q}(f))<1$.) Since $s\leq p$, we get from above and from 5.4.16 that

5.4.23 $$\|y-y^s\| \leq Kpd(\bar{x}^s,D_{k_s}(f))^{\nu^{m_s}\beta_s(\nu)}.$$

Next, we want to estimate $\|x-\hat{x}^s\|$. We have that $\|x-\hat{x}^s\| \leq \sum_{q=1}^{s} \|\hat{x}^{q-1}-\hat{x}^q\|$ (where $\hat{x}^0 = x$). Now each component of $\hat{x}^{q-1}-\hat{x}^q$ is by definition a component of $\tilde{x}^{q-1}-x^q$ (where we also define $\tilde{x}^0 = x$). Since we have at most p components, it follows from 5.4.13 that $\|\hat{x}^{q-1}-\hat{x}^q\| \leq p(p+1)d(\bar{x}^q,D_{k_q}(f))^{\nu^{m_q}\beta_q(\nu)}$. Arguing as in the case of the estimate of $\|y-y^s\|$, we get that

5.4.24 $$\|x-\hat{x}^s\| \leq p^2(p+1)d(\bar{x}^s,D_{k_s}(f))^{\nu^{m_s}\beta_s(\nu)}.$$

Now put $\hat{x} = \hat{x}^s$, $k = k_s$, $y' = y^s$, $x' = \bar{x}^s$, $\beta = \beta_s$, $\tilde{\beta} = \tilde{\beta}_s$ and $m = t_s \deg \beta_s$.

Suppose that it is also possible to for each $q = 1,\ldots,s$ to obtain polynomials such that also 5.4.22 holds. Then it will follow from 5.4.14, 5.4.15, 5.5.23 and 5.5.24 that (1), (2), (3) of the lemma hold. (In 5.5.23 we also have to redefine K such that $K>p^2(p+1)$ if necessary.)

To see that we can satisfy 5.4.12 and 5.4.22, we reason as follows: Recall the integer valued function $u(t)$, and the integer d from Lemma 5.3. Choose $t_p = t$ and then $t_{p-1},\dots,t_1$ recursively such that we always have $t_{q-1} = (t_q+d+1)u(t_q)(d+1)+1$. Also choose $g_1 = g$. We wish to satisfy the inequalities 5.4.12 and 5.4.22 at each step. We must therefore have:

$$5.4.12 \qquad \nu^{m_q}\beta_q(\nu)-1 > 2\beta_{q-1}(\nu)\beta_{q-1}(\nu), \qquad q = 2,\dots,s$$

and

$$5.4.22 \qquad \nu^{m_{q-1}}\beta_{q-1}(\nu) \geq 2\lambda\nu^{m_q}\beta_q(\nu)\tilde{\beta}_q(\nu), \quad q = 2,\dots,s.$$

Since g_1 and t_1 now is chosen, we can apply 5.3, and we get the point $\bar{x}^1$, and we also get the polynomials $\beta_1(\nu)$, $\tilde{\beta}_1(\nu)$, and the integer $m_1 = t_1\deg\beta_1$. If $\ell_1 = k_1$, then we stop at this step. If $\ell_1>k_1$, we put $g_2 = \tilde{\beta}_1\beta_1$. Now, t_2, g_2 are defined, and we can proceed and apply 5.3 once more. We will then get the point $\bar{x}^2$, and the polynomials β_2, $\tilde{\beta}_2$, and the integer $m_2 = t_2\deg\beta_2$. Now we want to see that there is an M, only dependent of g and f, such that $\nu>M$ implies that 5.4.12 and 5.4.22 hold when $q = 2$.

First, let us estimate the degree of the right and left hand side of 5.4.12 and 5.4.22. From Lemma 5.3 follows that the degree of the left hand side of 5.4.12 is $(t_2+1)\deg\beta_2\geq(t_2+1)\deg g_2 = (t_2+1)(\deg\tilde{\beta}_1+\deg\beta_1)$. The degree of the right hand side of 5.4.12 is exactly $\deg\tilde{\beta}_1+\deg\beta_1$. Since $u(t)>1$ for all positive t, we get from the definition of t_2 that $t_2+1\geq 2$. It follows that the degree of the left hand side of 5.4.12 is larger than the degree of the right hand side.

The degree of the left hand side of 5.4.22 is $(t_1+1)\deg\beta_1$. The degree of the right hand side is $(t_2+1)\deg\beta_2+\deg\tilde{\beta}_2\leq (t_2+1+d)\deg\beta_2\leq(t_2+1+d)u(t_2)\deg g_2 = (t_2+1+d)u(t_2)(\deg\beta_1+\deg\tilde{\beta}_1) \leq(t_2+1+d)u(t_2)(d+1)\deg\beta_1$. Since by definition

$t_1>(t_2+1+d)u(t_2)(d+1)$, the left hand side of 5.4.22 has degree larger than the right hand side. Now from 5.3 also follows that the number of non-zero terms, and the coefficients of β_1 and $\tilde{\beta}_1$ have upper bounds depending on the number of the non-zero terms and upper bounds of the coefficients in $g_1 = g$, and also on upper bounds of the derivatives of f, and on the constants and exponents in $L_1(f),\ldots,L_{p+1}(f)$. Since $g_2 = \tilde{\beta}_1\beta_1$, it will also follow from 5.3, that upper bounds of the coefficients and number of terms of β_2, $\tilde{\beta}_s$ and $\nu^{m_2}\beta_2(\nu)$ also are dependent on all this. Also the β_1, β_2, $\tilde{\beta}_1$, $\tilde{\beta}_2$ can be chosen to have all coefficients larger than 1. From this follows that there is an M only dependent on g and f such that $\nu>M$ implies that 5.4.12 and 5.4.22 hold provided the left hand side has larger degree than the right hand side. So $\nu>M$ implies that 5.4.12 and 5.4.22 hold when $q = 2$.

Now if $\ell_2 = k_2$, we stop our approximation procedure. If $\ell_2>k_2$, we put $g_3 = \beta_2\tilde{\beta}_2$, and we can argue as above to get 5.4.12 and 5.4.22 to hold when $q = 3$. Since $t_2>(t_3+1+d)u(t_3)(d+1)$, we get the degree of the left hand side of 5.4.12 and 5.4.22 to be larger of the right hand side when $q = 3$. Making M somewhat larger (if necessary), and letting $\nu>M$, we get 5.4.12 and 5.4.22 to hold when $q = 3$, Repeating the above argumentation, we get 5.4.12 and 5.4.22 to hold for $q = 1,\ldots,s$.

Using that $\deg\beta_q \leq u(t_q)\deg g_q$, $\deg\tilde{\beta}_q \leq d\deg\beta_q \leq du(t_q)\deg g_q$, that $g_q = \beta_{q-1}\tilde{\beta}_{q-1}$, and using the recursive definition of t_{q-1}, and the fact that $t_p = t$, $g_1 = g$, we will find that we can redefine u such that $\deg\beta_s \leq u(t)\deg g$ always holds (independent of for what s our approximation procedure stops). Also the other claims on $\beta_s = \beta$, $\tilde{\beta}_s = \tilde{\beta}$ follow easily from our repeatedly use of 5.3, and from corresponding statements in 5.3. This will also be the case of the claims on the neighbourhood U. Also claims on M follow easily from the above arguments, and claims of K follow from the proof of 5.5.23. This completes the proof of Lemma 5.4.

The next lemma will be the final one in this section. In this lemma we will show that if f satisfies $L_1(f),\dots,L_{p+1}(f)$ then any singular value can be approximated arbitrarily good by points satisfying condition (1), (2), (3) and (5) of 4.2. The lemma will thus be the main tool in the proof of 4.2.

<u>Lemma 5.5</u>. Let $f:(R^n,0) \to (R^p,0)$ satisfy $L_1(f),\dots,L_{p+1}(f)$. Let t be a positive integer, and let g be a positive polynomial in one variable. Then there exists a representative of f (which we also denote f) defined on some neighbourhood U, and there exist constants $M, K, C, C'>1$, such that the following holds:

Let $y\in f(\Sigma(f))-\{0\}$, $f^{-1}(y)\cap\Sigma(f) = \{x_1,\dots,x_\ell\}$ and $x = (x_1,\dots,x_\ell)\in(R^n)^\ell$. Let $\nu>M$. Then there exists another point $y'\in f(\Sigma(f))-\{0\}$, and there exist positive polynomials $\beta(\nu)$, $\tilde\beta(\nu)$, $\gamma(\nu)$, $\tilde\gamma(\nu)$, $\varepsilon(\nu)$ satisfying the following:

(1) Let $S' = \{x_1',\dots,x_k'\} = f^{-1}(y')\cap\Sigma(f)$, and let $x' = (x_1',\dots,x_k')\in(R^n)^k$. Then we have:

$$d(x',D_k(f)) \geq \min_{1\leq i\leq k} \|x_i'\|^{\tilde\beta(\nu)}.$$

(2) Let $B = B_f^k(S')$ be the linear map defined in §2, and suppose that $r = \operatorname{corank} B$. Let $b = b_f^r(S')$. Then we have:

$$b \geq d(x',D_k(f))^{\beta(\nu)}.$$

(3) Let $r>0$, and let $U_{x_i'}(r)$ or $U_{y'}(r)$ denote the ball at x_i' or y' with radius r. Put $U_{S'}(r) = \bigcup_{i=1}^{k} U_{x_i'}(r)$. Let $r = (1/C)(dx',D_k(f))^{\gamma(\nu)}$, $r' = (1/C')d(x',D_k(f))^{\gamma(\nu)}$ and $r'' = (1/C')d(x',D_k(f))^{\bar{\gamma}(\nu)}$.
Then the $U_i(r)$'s above are disjoint, and $f(U_{S'}(r)) \subset U_{y'}(r')$. Further, let $U_{S'}$ and $U_{y'}$ be the neighbourhoods at S' and y' which are assosiated to the map σ of Proposition 2.4 (since f satisfies $L_1(f),\ldots,L_p(f)$, f is stable at S' so σ exists). Then $U_{S'}(r) \subset U_{S'}$ and $U_{y'}(r') \subset U_{y'}$.

(4) There exist a integer $m>1$ and a point $\check{x} = (\check{x}_1,\ldots,\check{x}_\ell)$ with components among the points in S' such that:

$$\|x-\check{x}\| \leqslant Kd(x',D_k(f))^{\nu^m\beta(\nu)},$$
$$\|y-y'\| \leqslant Kd(x',D_k(f))^{\nu^m\beta(\nu)}$$

where $m \geqslant t \deg \beta(\nu)$.

(5) Consider $U_S(r/2)$ and $U_{y'}(r'')$. Suppose that $x \in f^{-1}(U_{y'}(r'')) \cap U_S(r/2)$. Then we have

$$J(f)(x) \geqslant \|x\|^{\varepsilon(\nu)}.$$

Further, we always have $\nu^{m-1}\beta(\nu) > \bar{\gamma}(\nu) > 2\gamma(\nu) > \gamma(\nu) > \beta(\nu)$. β, $\bar{\gamma}$, γ, ε will be dependent of the particular point y, but can be chosen among a finite number of candidates. We also have: There exists an integer valued function $u = u(t)$ such that $u(t)\deg g \geqslant \deg \nu^m\beta(\nu)$, $\deg \beta$, $\deg \tilde{\beta}$, $\deg \bar{\gamma}$, $\deg \gamma$, $\deg \varepsilon \geqslant \deg g$.

Further, u, upper bounds and lower bounds of the coefficients of β, $\tilde{\beta}$, γ, $\hat{\gamma}$, ε are of the same type as the corresponding bounds in 5.3 and 5.4. Also U and M are dependent of f in a way that corresponds to statements in 5.4. At last, K, C, C' are only dependent on bounds of the derivatives of f up to a certain order.

Proof of Lemma 5.5. First choose positive polynomials g_j and integers t_j, $j = 1,\dots,p$. We will use each g_j, t_j in the hypothesis of 5.4 which we will apply at most p times. Let $\nu > M$ be a constant, where M is the M of 5.4, and suppose that for each j we have given positive polynomials $\gamma_j(\nu)$, $\bar{\gamma}_j(\nu)$, $\varepsilon_j(\nu)$, $\tau_j(\nu)$. To make this proof work, all these polynomials and integers have to be defined carefully dependent of each other. We will however explain this dependence later on, and at the moment only suppose that the polynomials are given.

Let us start the proof by choosing a representative of f such that the conclusion of 5.4 is valid for all g_j, t_j. Let $y \in f(\Sigma(f)) - \{0\}$, and let $f^{-1}(y) \cap \Sigma(f) = \{x_1,\dots,x_\ell\}$, $x = (x_1,\dots,x_\ell)$. Use 5.4 to find a point y^1 and polynomials β_1, $\tilde{\beta}_1$ satisfying the conclusion of 5.4. Let $S_1 = \{x_1^1,\dots,x_{k_1}^1\} = f^{-1}(y^1) \cap \Sigma(f)$, and let $x^1 = (x_1^1,\dots,x_{k_1}^1)$. Put $r_1 = \operatorname{corank} B_f^{k_1}(S_1)$, and $b_1 = b_f^{r_1}(S_1)$. Then we have that

5.5.1 $\|y-y^1\| \leq Kd(x^1,D_{k_1}(f))^{\nu^{m_1}\beta_1(\nu)}$ where $m_1 \geq t_1 \deg \beta_1$,

5.5.2 $b_1 \geq d(x^1,D_{k_1}(f))^{\beta_1(\nu)}$.

Recall Proposition 2.4. In this proposition we defined a right inverse σ to $tf+\omega f$. We had that $\sigma: V(f|U_{S_1}) \to V(U_{S_1}) \times (V(U_{y^1})$ where U_{S_1}, U_{y^1} were a union of ball centered at each x_i^1 and a ball at y^1, with radius $(1/C_f)(a_1 b_1)^{\beta}$ and $(1/C'_f)(a_1 b_1)^{\beta}$ respectively. Here C_f, C'_f are constants depending on bounds of the derivatives of f, and β is independent of f. The C_f, C'_f can thus be made independent of the particular set S_1. a_1 is here $a_f^{k_1}(x^1)$. Now, since the symbol β already appears in the text of Lemma 5.5, we will (to prevent ambiguity) in this proof denote the β of 2.4 by ρ. We wish to find conditions on $\gamma_1(\nu)$ for (3) to hold when we consider the point y^1 and set S_1. Since f satisfies $L_1(f),\ldots,L_{p+1}(f)$, we have

5.5.3 $$a_1 \geq d(x^1,D_{k_1}(f))^{\alpha_{k_1}}.$$

(By choosing U small enough, and making the exponent larger, we can always suppose that the constant in $L_{k_1}(f)$ can be taken to be 1.) 5.5.2 and 5.5.3 imply that

5.5.4 $$(a_1 b_1)^{\rho} \geq d(x^1,D_{k_1}(f))^{\rho(\beta_1(\nu)+\alpha_{k_1})}.$$

Let $C = C_f$, $C' = C'_f$. The C_f, C'_f of 2.4 can easily be taken to be such that $|z_1-z_2|<(1/C_f)r \Rightarrow |f(z_1)-f(z_2)|<(1/C'_f)r$ for any r and z_1, z_2. Suppose that

5.5.5 $$\gamma_1(\nu) \geq \rho(\beta_1(\nu)+\alpha_{k_1}).$$

Then we get that $U_{S_1}(r) \subset U_{S_1}$, $U_{y^1}(r') \subset U_{y^1}$. (Provided $d(x^1,D_{k_1}(f))<1$).). We also get that $f(U_{S_1}(r)) \subseteq U_{y^1}(r')$. It follows that condition (3) of this Lemma holds in this case. Suppose that $\bar{\gamma}_1(\nu)>2\gamma_1(\nu)$, and define $U_{y^1}(r'')$ in terms of such a $\bar{\gamma}_1$. Suppose there exists $\bar{z}^1 \in f^{-1}(U_{y^1}(r''))\cap \sim U_{S_1}(r/2)$ such that $J(f)(\bar{z}^1)<\|\bar{z}^1\|^{\varepsilon_1(\nu)}$. Then we have to refine the situation such that (5) holds. Put $\bar{y}^1 = f(\bar{z}^1)\in U_{y^1}(r'')$. Put $\bar{x}^1 = (\bar{x}^1_1,\ldots,\bar{x}^1_{k_1+1})$ where $\bar{x}^1_i = x^1_i$ for $i<k_1$, and $\bar{x}^1_{k_1+1} = \bar{z}^1$. Then we have:

5.5.6 $$\sum_{1\leq i,j\leq k_1+1} \|f(\bar{x}_i^1)-f(\bar{x}_j^1)\|^2 + \sum_{1\leq i\leq k_1+1} J(f)(\bar{x}_i) \leq k_1(1/C')d(x^1,D_{k_1}(f))^{\bar{\gamma}_1(\nu)} + \|\bar{z}^1\|^{\varepsilon_1(\nu)}.$$

(Recall that $f(\bar{x}_i^1) = f(\bar{x}_j^1)$ for $i,j\leq k_1$, and $\bar{x}_i^1\in\Sigma(f)$ for $i\leq k_1$). Now we will need to compare $d(x^1,D_{k_1}(f))$ with $d(\bar{x}^1,D_{k_1+1}(f))$. Since f satisfies $L_1(f),\ldots,L_{p+1}(f)$, f is ∞-$\mathcal{K}$ determined. This implies that there exists a λ such that $x\in\Sigma(f)$ implies that $\|f(x)\|\geq\|x\|^{\lambda}$. (Sublemma 4.4.) Let us suppose that $C'>1$, then we get that

5.5.7 $$\|\bar{y}^1\| \geq \|y^1\|-(1/C')d(x^1,d_{k_1}(f))^{\bar{\gamma}_1(\nu)} \geq \|y^1\|-(\min_j\|x_j^1\|)^{\bar{\gamma}_1(\nu)}.$$

Suppose that we always have $\bar{\gamma}_1(\nu)>2\lambda$, then we get that $\|x_j^1\|^{\bar{\gamma}_1(\nu)}\leq\|y^1\|^2$. Suppose that $\|y^1\|<\frac{1}{2}$, then we get that

5.5.8 $$\|\bar{y}^1\| \geq \|y^1\|-\|y^1\|^2 \geq \tfrac{1}{2}\|y^1\| \geq \|y^1\|^2.$$

Also since Df is bounded, we can show that we always have $\|f(x)\|^2\leq\|x\|$. This implies that

5.5.9 $$\|\bar{z}^1\| \geq \|\bar{y}^1\|^2 \geq \|y^1\|^4 \geq \min_{1\leq j\leq k_1} \|x_j^1\|^{4\lambda} \geq d(x^1,D_{k_1}(f))^{4\lambda}.$$

From the definition of $d(\bar{x}^1,D_{k_1+1}(f))$ and of $\bar{x}^1$, it follows that we either have $d(\bar{x}^1,D_{k_1+1}(f)) = d(x^1,D_{k_1}(f))$ or $d(\bar{x}^1,D_{k_1+1}(f))\geq \min_{1\leq i\leq k_1} \|x_i^1-\bar{z}^1\|$ or $d(\bar{x}^1,D_{k_1+1}(f)) = \|\bar{z}^1\|$. So either $d(\bar{x}^1,D_{k_1+1}(f)) = d(x^1,D_{k_1}(f))$ or $d(\bar{x},D_{k_1+1}(f))\geq(1/(2C))d(x^1,D_{k_1}(f))^{\gamma_1(\nu)}$ (since $\bar{z}^1\in{\sim}U_{S_1}(r/2)$) or $d(\bar{x}^1,D_{k_1+1}(f))\geq d(x^1,D_{k_1}(f))^{4\lambda}$ (from 5.5.9). Let us suppose that $d(x^1,D_{k_1}(f))<1/(2C)$, and that always $\gamma_1(\nu)>4\lambda$, then we get that in any case:

5.5.10 $$d(\bar{x}^1,D_{k_1+1}(f)) \geq d(x^1,D_{k_1}(f))^{\gamma_1(\nu)+1}.$$

Again using that f is ∞-$\mathcal{K}$ determined, we have that for all x

5.5.11 $\|f(x)\| + J(f)(x) \geq \|x\|^{\lambda}$ for a suitble λ.

Suppose that we always have $\varepsilon_1(\nu) > \lambda+1$, then we get from 5.5.11 that

5.5.12 $\|\bar{y}^1\| = \|f(\bar{z}^1)\| \geq \|\bar{z}^1\|^{\lambda} - \|\bar{z}^1\|^{\lambda+1} \geq \frac{1}{2}\|\bar{z}^1\|^{\lambda} \geq \|\bar{z}^1\|^{\lambda+1}$.

(Provided $\|\bar{z}^1\| < \frac{1}{2}$.) Arguing as in 5.5.8, but using an opposite triangle inequality, we get that $\|\bar{y}^1\|^2 \leq \|y^1\|$. Using this, and 5.5.12, we get that

5.5.13 $\|\bar{z}^1\| \leq \|\bar{y}^1\|^{1/(\lambda+1)} \leq \|y^1\|^{1/(2(\lambda+1))} \leq (\min_j \|x_j^1\|)^{1/(4(\lambda+1))}$.

(Since we always have that $\|y^1\|^2 \leq \|x_j^1\|$ in a neighbourhood of 0.) Now from the conclusion of 5.4 we had that

5.5.14 $$d(x^1, D_{k_1}(f)) \geq \min_j \|x_j^1\|^{\hat{\beta}_1(\nu)}.$$

This and 5.5.13 imply that

5.5.15 $$\|\bar{z}^1\| \leq d(x^1, D_{k_1}(f))^{1/(4(\lambda+1)\hat{\beta}_1(\nu))}.$$

Using 5.5.10 and 5.5.15 in 5.5.6 we get that

5.5.16 $$\sum_{1\leq i,j\leq k_1+1} \|f(\bar{x}_i^1) - f(\bar{x}_j^1)\|^2 + \sum_{1\leq i\leq k_1+1} J(f)(\bar{x}_i)$$
$$\leq (k_1/C')d(\bar{x}^1, D_{k_1+1}(f))^{\bar{\gamma}_1(\nu)/(\gamma_1(\nu)+1)}$$
$$+ d(\bar{x}^1, D_{k_1+1}(f))^{\varepsilon_1(\nu)/(4(\lambda+1)(\gamma_1(\nu)+1)\hat{\beta}_1(\nu))}.$$

In this situation we want to apply Lemma 3.8. In the hypothesis of 3.8 V will be the set of singular multijets in $(R^p \times J^p(n,p))^{k_1+1}$ with common target point. We will put $C = \frac{1}{2}$ (where now C is the C of 3.8), and let the α of 3.8 be the $\tau_1(\nu)$ we had in the beginning of the proof. We must therefore suppose that $\tau_1(\nu)>1$. With this set V it is easy to see that the polynomials h_i, that generates $I(V)$, can be chosen such that $\sum_{i,j} \|f(\bar{x}_i^1)-f(\bar{x}_j^1)\|^2+\sum_i J(f)(\bar{x}_i)$ corresponds to $\sum_i \|\bar{h}_i(\bar{x}^1)\|^2$. (Recall the notation of Lemma 3.8.) Define the neighbourhood $N_1 = N_V(\frac{1}{2},\tau_1(\nu),f)$ of 3.8. Let $\bar{K}$, $\bar{\beta}$, P be the constants and the polynomial of the conclusion of 3.8 in this case. Also suppose that our neighbourhood U is defined such that the conclusion of 3.8 is valid. We now get from the conclusion of 3.8 that

$\sum_{i,j} \|f(\bar{x}_i^1)-f(\bar{x}_j^1)\|^2+\sum_i J(f)(\bar{x}_i^1)\leq(1/\bar{K})(\frac{1}{2})^{\bar{\beta}}d(\bar{x}^1,D_{k_1}(f))^{P(\tau_1(\nu))}$ implies that $\bar{x}^1\in N_1$. Suppose U is so small that $d(\bar{x}^1,D_{k_1+1}(f))\leq\min(\frac{1}{2},1/(2\bar{K}))$. It is also easy to see that the C' can be chosen such that $C'>p$, so $k_1/C'\leq 1$. Let us suppose that $\bar{\gamma}_1(\nu)$, $\gamma_1(\nu)$, $\varepsilon_1(\nu)$ can be chosen such that

5.5.17 $$\bar{\gamma}_1(\nu) \geq (\gamma_1(\nu)+1)(P(\tau_1(\nu))+\bar{\beta}+1)$$

5.5.18 $$\varepsilon_1(\nu) \geq 4(\lambda+1)(\gamma_1(\nu)+1)\hat{\beta}_1(\nu)(P(\tau_1(\nu))+\bar{\beta}+1).$$

Then 5.5.17 implies that

5.5.19 $$(k_1/C')d(\bar{x}^1,D_{k_1+1}(f))^{\bar{\gamma}_1(\nu)/(\gamma_1(\nu)+1)}$$
$$\leq (1/(2\bar{K}))(\tfrac{1}{2})^{\bar{\beta}}d(\bar{x}^1,D_{k_1+1}(f))^{P(\tau_1(\nu))},$$

and 5.5.18 implies that

5.5.20 $$d(\bar{x}^1,D_{k_1+1}(f))^{\varepsilon_1(\nu)/(4(\lambda+1)(\gamma_1(\nu)+1)\hat{\beta}_1(\nu))}$$
$$\leq (1/(2\bar{K}))(\tfrac{1}{2})^{\bar{\beta}}d(\bar{x}^1,D_{k_1+1}(f))^{P(\tau_1(\nu))}.$$

From 5.5.16 and the above, we get that $\bar{x}^1\in N_1$, provided 5.5.17 and 5.5.18 hold.

Let us suppose this, then we get that there exists a point $\tilde{x}^1 = (\tilde{x}^1_1, \ldots, \tilde{x}^1_{k_1+1}) \in (U)^{k_1+1}$ with all component singular points with common image point such that

5.5.21 $$\|\bar{x}^1 - \tilde{x}^1\| \leq \tfrac{1}{2} d(\tilde{x}^1, D_{k_1+1}(f))^{\tau_1(\nu)}.$$

Put $\tilde{y}^1 = f(\tilde{x}^1_1) = \ldots = \tilde{f}(\tilde{x}^1_{k_1+1})$. Let us suppose that U is small, and that $\tau_1(\nu) > 1$, then it is easy to deduce from 5.5.19 using the triangle inequality that

5.5.22 $$d(\bar{x}^1, D_{k_1+1}(f))^2 \leq d(\tilde{x}^1, D_{k_1+1}(f)),\ \|\bar{x}^1_i\|^2 \leq \|\tilde{x}^1_i\|,\ 1 \leq i \leq k_1+1,$$

5.5.23 $$d(\tilde{x}^1, D_{k_1+1}(f))^2 \leq d(\bar{x}^1, D_{k_1+1}(f)),\ \|\tilde{x}^1_i\|^2 \leq \|\bar{x}^1_i\|,\ 1 \leq i \leq k_1+1.$$

In this situation we will apply Lemma 5.4 once more with g_2, t_2 in the hypothesis of 5.4, and we approximate $\tilde{y}^1$ with a point y^2 in the target such that the conclusion of 5.4 holds. Put $S_2 = f^{-1}(y^2) \cap \Sigma(f) = \{x^2_1, \ldots, x^2_{k_2}\}$, $x^2 = (x^2_1, \ldots, x^2_{k_2})$ $r_2 = \text{corank } B^{k_2}_f(S_2)$, and $b_2 = b^{r_2}_f(S_2)$. Then we find polynomials $\beta_2(\nu)$, $\tilde{\beta}_2(\nu)$ and an integer m_2, such that $m_2 \geq t_2 \deg g_2$, satisfying the conclusion of 5.4. We thus get

5.5.24 $$\|\tilde{y}^1 - y^2\| \leq K d(x^2, D_{k_2}(f))^{\nu^{m_2}\beta_2(\nu)}.$$

From 5.4 (3) we can find a point $\hat{x}^2 = (\hat{x}^2_1, \ldots, \hat{x}^2_{k_1+1})$ with components among the points in S_2 such that

5.5.25 $$\|\tilde{x}^1 - \hat{x}^2\| \leq K d(x^2, D_{k_2}(f))^{\nu^{m_2}\beta_2(\nu)}.$$

Also when we approximated y with y^1, applying 5.4 in the first case, we had another point $\hat{x}^1 = (\hat{x}^1_1, \ldots, \hat{x}^1_\ell)$ with components among the poins in S_1 such that

5.5.26 $$\|x - \hat{x}^1\| \leq K d(x^1, D_{k_1}(f))^{\nu^{m_1}\beta_1(\nu)}.$$

Define $\check{x}^1 = (\check{x}_1^1,\ldots,\check{x}_\ell^1) = \hat{x}^1$, and define $\check{x}^2 = (\check{x}_1^2,\ldots,\check{x}_\ell^2)$ by $\check{x}_i^2 = \hat{x}_j^2 \Leftrightarrow \check{x}_i^1 = \bar{x}_j^1$. (This make sense since $\hat{x}^2$ has $k_1+1>k_1$ components, and by 5.4 $\{\check{x}_1^1,\ldots,\check{x}_\ell^1\} \subset S_1$.) Now we want to compare k_2 and k_1. We will need to compare $d(\tilde{x}^1,D_{k_1+1}(f))$ and $d(x^1,D_{k_1}(f))$. Using 5.5.15 and 5.5.23, we get that

5.5.27 $$d(\tilde{x}^1,D_{k_1+1}(f)) \leq d(\bar{x}^1,D_{k_1+1}(f))^{\frac{1}{2}} \leq \|\bar{z}^1\|^{\frac{1}{2}} \leq d(x^1,D_{k_1}(f))^{1/(8(\lambda+1)\tilde{\beta}_1(\nu))}.$$

We will also need to compare $d(x^1,D_{k_1}(f))$ and $d(x^2,D_{k_2}(f))$. Using 5.5.25, we easily deduce that for U sufficiently small we have:

5.5.28 $$\|\hat{x}_i^2\|^2 \leq \|\tilde{x}_i^1\|, \quad \|\tilde{x}_i^1\|^2 \leq \|\hat{x}_i^2\| \quad \text{for } i = 1,\ldots,k_1+1.$$

From this and from 5.5.23 and 5.5.14 we get that

5.5.29 $$d(x^2,D_{k_2}(f)) \leq \min_{1\leq i\leq k_2} \|x_i^2\| \leq \min_{1\leq j\leq k_1+1} \|\hat{x}_j^2\| \leq \min_{1\leq j\leq k_1+1} \|\tilde{x}_j^1\|^{\frac{1}{2}} \leq \min_{1\leq j\leq k_1+1} \|\bar{x}_j^1\|^{1/4} \leq \min_{1\leq j\leq k_1} \|x_j^1\|^{1/4} \leq d(x^1,D_{k_1}(f))^{1/(4\tilde{\beta}_1(\nu))}.$$

From 5.5.21, 5.5.25, 5.5.27 and 5.5.29 we now get that

5.5.30 $$\|\bar{x}^1-\hat{x}^2\| \leq (\tfrac{1}{2})d(x^1,D_{k_1}(f))^{\tau_1(\nu)/(8(\lambda+1)\tilde{\beta}_1(\nu))} + Kd(x^1,D_{k_1}(f))^{\nu^{m_2}\beta_2(\nu)/(4\tilde{\beta}_1(\nu))}.$$

Recall that we had $\bar{x}^1 = (x_1^1,\ldots,x_{k_1}^1,\bar{z}^1)$. Also note that we get that the balls $U_{x_i^1}(r)$ all are disjoint provided 5.5.5 holds. (This follows, since U_{S^1} consists of a union of disjoint balls, and each $U_{x_1^1}(r)$ is contained in such a ball provided 5.5.5 holds.) We also had that $\bar{z}^1 \in \sim U_{S_1}(r/2)$.

From this we also get that the balls at $\bar{x}_i$, $i = 1,\dots,k_1+1$ with radius $r/4$ are disjoint. Supposing that $d(x^1,D_{k_1}(f))<1/(4C)$ the balls will still be disjoint if we take the radius to be $d(x^1,D_{k_1}(f))^{\gamma_1(\nu)+1}$. (Recall the definition of r.) Let us now suppose that

5.5.31 $$\nu^{m_2\beta_2(\nu)} \geq 4\beta_1(\nu)(\gamma_1(\nu)+2)$$

5.5.32 $$\tau_1(\nu) \geq 8(\lambda+1)\tilde{\beta}_1(\nu)(\gamma_1(\nu)+1).$$

Let us also suppose that $d(x^1,D_{k_1}(f))\leq 1/(2K)$. Then we get from 5.5.30 that

5.5.33 $$\|\bar{x}^1-\hat{x}^2\| \leq d(x^1,D_{k_1}(f))^{\gamma_1(\nu)+1}.$$

It follows that each $\hat{x}_i^2$ belongs to the ball at $\bar{x}_i^1$ with radius $d(x^1,D_{k_1}(f))^{\gamma_1(\nu)+1}$. Since these balls are disjoint, no one of the components of $\hat{x}^2$ can coincide. Since $\{\hat{x}_1^2,\dots,\hat{x}_{k_1+1}^2\} \subseteq S_2 = \{x_1^2,\dots,x_{k_2}^2\}$, we get that $k_2\geq k_1+1$.

Next we will need to estimate $\|y-y^2\|$ and $\|x-\check{x}^2\|$. From 5.5.21, 5.5.24, 5.5.27, and the fact that $f(\bar{x}_i^1) = y^1$ for $i\leq k_1$, we get that

5.5.34 $$\begin{aligned}\|y^1-y^2\| &\leq K_1 d(\tilde{x}^1,D_{k_1+1}(f))^{\tau_1(\nu)} + Kd(x^2,D_{k_2}(f))^{\nu^m\beta_2(\nu)}\\ &\leq K_1 d(x^1,D_{k_1}(f))^{\tau_1(\nu)/(8(\lambda+1)\tilde{\beta}_1(\nu))}\\ &\quad + Kd(x^2,D_{k_2}(f))^{\nu^m\beta_2(\nu)}.\end{aligned}$$

In 5.5.34 K_1 is a constant bounding the derivatives of f.

Let us compare $d(x^2,D_{k_2}(f))$ and $d(x^1,D_{k_1}(f))$ in the other direction. Using 4.4, we have that $\|x_i^2\|\geq\|x_j^2\|^{\lambda+1}$ for all i,j. Here λ is the exponent of the Lojasiewicz inequality 5.5.11 we get from ∞-$\mathcal{K}$ determinacy. Using this, 5.5.9, 5.5.22, 5.5.28 and the conclusion of 5.4 in the second approximation, we get that

5.5.35 $$d(x^2,D_{k_2}(f)) \geq \min_{1\leq i\leq k_2} \|x_i^2\|^{\tilde{\beta}_2(\nu)} \geq \min_{1\leq j\leq k_1+1} \|\hat{x}_j^2\|^{(\lambda+1)\tilde{\beta}_2(\nu)}$$

$$\geq \min_{1\leq i\leq k_1+1} \|\tilde{x}_i^1\|^{2(\lambda+1)\tilde{\beta}_2(\nu)} \geq \min_{1\leq i\leq k_1+1} \|\bar{x}_i^1\|^{4(\lambda+1)\tilde{\beta}_2(\nu)}$$

$$\geq \min_{1\leq i\leq k_1} \|x_i^1\|^{16\lambda(\lambda+1)\tilde{\beta}_2(\nu)} \geq d(x^1,D_{k_1}(f))^{16\lambda(\lambda+1)\tilde{\beta}_2(\nu)}.$$

Now let us suppose that

5.5.36 $$\tau_1(\nu) \geq (\nu^{m_2\beta_2(\nu)+1})(8(\lambda+1)\tilde{\beta}_1(\nu))(16\lambda(\lambda+1)\tilde{\beta}_2(\nu)).$$

Let us also suppose that $d(x^1,D_{k_1}(f))<1/K_1$. Then 5.5.34, 5.5.35, and 5.5.36 imply that

5.5.37 $$\|y^1-y^2\| \leq (K+1)d(x^2,D_{k_2}(f))^{\nu^{m_2\beta_2(\nu)}}.$$

We also suppose that

5.5.38 $$\nu^{m_1\beta_1(\nu)} \geq 16\lambda(\lambda+1)\tilde{\beta}_2(\nu)^{\nu^{m_2\beta_2(\nu)}}.$$

Then 5.5.1 and 5.5.35 imply that

5.5.39 $$\|y-y^1\| \leq Kd(x^2,D_{k_2}(f))^{\nu^{m_2\beta_2(\nu)}}.$$

Now 5.5.37 and 5.5.39 imply that

5.5.40 $$\|y-y^2\| \leq (2K+1)d(x^2,D_{k_2}(f))^{\nu^{m_2\beta_2(\nu)}}.$$

We also need to estimate $\|x-\check{x}^2\|$. Using the definition of $\check{x}^1$ and $\check{x}^2$, it is clear that each component of $\check{x}^1-\check{x}^2$ is a component of $\bar{x}^1-\hat{x}^2$. Using 5.5.21, 5.5.25, 5.5.26, 5.5.27, 5.5.35, 5.5.36, and 5.5.38, we will obtain that

5.5.41 $$\|x-\check{x}^2\| \leq \ell(2K+1)d(x^2,D_{k_2}(f))^{\nu^{m_2\beta_2(\nu)}}.$$

Now we proceed by considering $\gamma_2(\nu)$, $\bar{\gamma}_2(\nu)$, and we use $\gamma_2(\nu)$, $\bar{\gamma}_2(\nu)$ to define $r = (1/C)d(x^2,D_{k_2}(f))^{\gamma_2(\nu)}$,

$r' = (1/C')d(x^2, D_{k_2}(f))^{\gamma_2(\nu)}$, $r'' = (1/C')d(x^2, D_{k_2}(f))^{\bar{\gamma}_2(\nu)}$. We put up inequalities that corresponds to 5.5.5 but involves $\gamma_2(\nu)$ and $\beta_2(\nu)$, such that if this inequality is satisfied, then we get (5) to hold when we consider the point set S_2. Now if $J(f)(z) \geq \|z\|^{\varepsilon_2(\nu)}$ for all $z \in f^{-1}(U_{y^2}(r'')) \cap \sim U_S(r/2)$, then we stop. Otherwize, we consider a point $\bar{z}^2 \in f^{-1}(U_{y^2}(r'')) \cap \sim U_S(r/2)$ such that $J(f)(\bar{z}^2) < \|\bar{z}^2\|^{\varepsilon_2(\nu)}$. Define $\bar{x}^2 = (x_1^2, \ldots, x_{k_2}^2, \bar{z}^2)$, and a neighbourhood N_2 by $\tau_2(\nu)$. Then we put up inequalities that corresponds to 5.5.17 and 5.5.18, but invole the index 2 instead of 1. If these inequalities are satisfied, then $\bar{x} \in N_2$, and we can find a point $\tilde{x}^2$ that corresponds to $\tilde{x}^1$ in the fist step. We define the image point $\tilde{y}^2$, and use Lemma 5.4 with t_3, g_3 in the hypothesis, and we approximate $\tilde{y}^2$ with a point y^3. We have $k_3 = f^{-1}(y_3) \cap \int(f)$. We put up inequalities that corresponds to 5.5.31 and 5.5.32. If these inequalities are satisfied, then we get $k_3 > k_2$. At 1st we define $\hat{x}^3$ and $\check{x}^3$ by the same sort of definitions as those we used when we defined $\hat{x}^2$ and $\check{x}^2$. Then we put up inequalities corresponding to 5.5.36 and 5.5.38 to get estimates of $\|y-y^3\|$ and $\|x-\check{x}^3\|$. Next we proceed starting with $\gamma_3(\nu)$, $\bar{\gamma}_3(3)$ etc. If all the inequalities that corresponds to 5.5.5, 5.5.17, 5.5.18, 5.5.31, 5.5.32, 5.5.36 and 5.5.38 always hold we get $k_1 < k_2 < k_3 < \ldots < k_j < k_{j+1} \ldots$. Since f is stable outside 0, and the set $S_1, S_2, \ldots, S_s$ we get defined consist of singular points, it is clear that the procedure have to stop for an $s \leq p$. If we can get all the inequalities to hold, we put $y' = y^s$, $S' = S_s$, $x' = x^s$, $\check{x} = \check{x}^s$, $\beta(\nu) = \beta_s(\nu)$, $\tilde{\beta}(\nu) = \beta_s(\nu)$, $m = m_s$, $\bar{\gamma}(\nu) = \bar{\gamma}_s(\nu)$, $\gamma(\nu) = \gamma_s(\nu)$, $\varepsilon(\nu) = \varepsilon_s(\nu)$. Since the procedure stops when $J(F)(z) \geq \|z\|^{\varepsilon_s(\nu)}$ for all $z \in f^{-1}(U_{y^s}(r'')) \cap \sim U_{S_s}(r/2)$, ($r$, r'' are defined in terms of γ_s, $\bar{\gamma}_s$), we get (5) of this Lemma to hold. (1), (2), (4) follows from the inequalities, at step s, that correspond to 5.5.14, 5.5.2, 5.5.40 and 5.5.41, and (3) follows from comments that correspond, a step s, to comments made for neighbourhoods $U_{S_1}(r)$ and $U_{y^1}(r')$.

Let us once again put up the crucial inequalities at a step that corresponds to an index $1 \leq j \leq p-1$.

5.5.5 $\quad \gamma_j(\nu) \geq \rho(\beta_j(\nu)+\alpha_{k_j})$,

5.5.17 $\quad \bar{\gamma}_j(\nu) \geq (\gamma_j(\nu)+1)(P(\tau_j(\nu))+\bar{\beta}+1)$,

5.5.18 $\quad \varepsilon_j(\nu) \geq 4(\lambda+1)(\gamma_j(\nu)+1)\tilde{\beta}_j(\nu)(P(\tau_j(\nu))+\bar{\beta}+1)$,

5.5.31 $\quad \nu^{m_{j+1}}\beta_{j+1}(\nu) \geq 4\tilde{\beta}_j(\nu)(\gamma_j(\nu)+2)$

5.5.32 $\quad \tau_j(\nu) \geq 8(\lambda+1)\tilde{\beta}_j(\nu)(\gamma_j(\nu)+1)$,

5.5.36 $\quad \tau_j(\nu) \geq (\nu^{m_{j+1}}\beta_{j+1}(\nu)+1)(8(\lambda+1)\tilde{\beta}_j(\nu))(16\lambda(\lambda+1)\tilde{\beta}_{j+1}(\nu))$,

5.5.38 $\quad \nu^{m_j}\beta_j(\nu) > 16\lambda(\lambda+1)\tilde{\beta}_{j+1}(\nu)\nu^{m_{j+1}}\beta_{j+1}(\nu)$.

In addition we want that

5.5.42 $\quad \nu^{m_j-1}\beta_j(\nu) > \bar{\gamma}_j(\nu) > 2\gamma_j(\nu) > \gamma_j(\nu) > \beta_j(\nu)$,

and also we someplaces in the proof suppose that our polynomials have some constant lower bounds.

Let us now explain how we can manage to get the above inequalities to hold. We will use arguments that corresponds to arguments proving similar inequalities in 5.4. First note that the polynomial P of 3.8 can always bet taken to be the same polynomial. Although we use 3.8 in different situations (V varies since k_j varies), we will only weaken the conclusion of 3.8 if a

P is replaced by some larger polynomial. We can thus find a P that works for all k_j. Let this P have degree α. Consider the integer d and integer valued function $u(t)$ which appear in the estimation of the degree of β and $\tilde{\beta}$ in 5.4. It is also clear that we can find a common d and a common $u(t)$ working for each index j. We choose $t_p, t_{p-1}, \ldots, t_1$ inductively by choosing $t_p = \max(t, d+1)$, and then choose
$t_{p-1} = 4(\alpha+1)u(t_p)(d+1), t_{p-2} = 4(\alpha+1)u(t_{p-1})(d+1), \ldots,$
$t_j = 4(\alpha+1)u(t_{j+1})(d+1), \ldots, t_1 = 4(\alpha+1)u(t_2)(d+1)$. Next choose $g_1 = g$. Let M be the constant of 5.4. Let $\nu > M$. Since g_1, t_1 are defined, we can use 5.4, and we get $\beta_1(\nu)$, $\tilde{\beta}_1(\nu)$ and m_1. Now we put $\gamma_1(\nu) = 2\rho(\beta_1(\nu)+\alpha_{k_1})$ so 5.5.5 holds for $j = 1$. We define $\tau_1(\nu) = 16(\lambda+1)\tilde{\beta}_1(\nu)(\gamma_1(\nu)+1)(\nu^{3(d+1)u(t_2)\deg \beta_{1+1}})$,
$\bar{\gamma}_1(\nu) = 2(\gamma_1(\nu)+1)(P(\tau_1(\nu))+\bar{\beta}+1)$ and
$\varepsilon_1(\nu) = 8(\lambda+1)(\gamma_1(\nu)+1)\tilde{\beta}_1(\nu)(P(\tau_1(\nu))+\bar{\beta}+1)$. So 5.5.17, 5.5.18, and 5.5.32 also hold.

Now we put $g_2(\nu) = \beta_1(\nu)(\gamma_1(\nu)+2)$. So we can use 5.4 once more to obtain $\beta_2(\nu)$, $\tilde{\beta}_2(\nu)$ and m_2. From 5.4 and our definitions it follows that upper bounds of the number of non-zero terms and the size of the coefficients of all polynomials with index 1 are only dependent of g (and of $L_1(f), \ldots, L_{p+1}(f)$ and bounds of the derivatives of f.) Also all the non-zero coefficients of these polynomials are larger than 1. From 5.4 it also follows that β_2, $\tilde{\beta}_2$ and $\nu^{m_2}\beta_2(\nu)$ have similar upper bounds which is however dependent of bounds of g_2. From the definition of g_2, we can find bounds of g_2 dependent of g. It follows that we also can find bounds of β_2, $\tilde{\beta}_2$, $\nu^{m_2}\beta_2(\nu)$ which also only are dependent of g. Let us now consider the remaining inequalities 5.5.31, 5.5.36 and 5.5.38 which when $j = 1$ also involve

β_2, $\tilde{\beta}_2$ and m_2. From the above remarks about the bounds of the coefficients of the polynomials on the right hand side of these inequalities, and from the fact that the polynomials of the left hand side have all their coefficients larger than 1 (in the case of 5.5.32 this follows from the definition of τ_1), it is clear that there is another constant M_1 only depending on g (and $L_1(f),\ldots,L_{p+1}(f)$ etc.) such that if $\nu > M_1$, and if the degree of the polynomials of the left hand side are greater than the degree of the polynomials of the right hand side in each of the inequalities 5.5.31, 5.5.36 and 5.5.38, then these inequalities actually also will hold themselves. Using 5.4 we will find that the degree of the left hand side of 5.5.31 has the lower bound $(t_2+1)\deg \beta_2 \geq (t_2+1)\deg g_2 = (t_2+1)(\deg \beta_1+\deg \gamma_1)$. By the definition of γ_1 we have $\deg \gamma_1 = \deg \beta_1$. So we have $(t_2+1)\deg \beta_2 \geq 2(t_2+1)\deg \beta_1$. The degree of the right hand side of 5.5.31 is by 5.4 $\deg \tilde{\beta}_1+\deg \gamma_1 \leq d \deg \beta_1+\deg \gamma_1 = (d+1)\deg \beta_1$. Since the u of 5.4 is ≥ 1, the inductive definition of t_2 implies that $t_2 \geq d+1$, so the left hand side of 5.5.31 has larger degree than the right hand side.

Next consider 5.5.36. By the definition of $\tau_1(\nu)$ the left hand side has degree $\deg \tilde{\beta}_1+\deg \gamma_1+3(d+1)u(t_2)\deg \beta_1$. The right hand side of 5.5.36 has degree $\deg (\nu^{m_2}\deg \beta_2)+\deg \tilde{\beta}_1+\deg \tilde{\beta}_2 \leq (d+1)u(t_2)\deg g_2+d \deg \beta_1 \leq (2(d+1)u(t_2)+d)\deg \beta_1 < (3(d+1))u(t_2)\deg \beta_1$. So the left hand side has degree greater than the right hand side.

At last consider 5.5.38. The left hand side has degree greater or equal $(t_1+1)\deg \beta_1$. The right hand side has degree $\deg \tilde{\beta}_2+\deg (\nu^{m_2}\beta_2(\nu)) \leq (d+1)u(t_2)\deg g_2 = 2(d+1)u(t_2)\deg \beta_1$. Now again $t_1 > 4(d+1)u(t_2)$, so the left hand side has degree greater than the right hand side. So we get the series of inequalities to hold provided $\nu > M_1$.

Let us also consider 5.5.42. By definition $\gamma_1 > \beta_1$. (We can always suppose that $\rho \geq 1$.) Also from the definition of $\bar{\gamma}_1$ follows that $\bar{\gamma}_1(\nu) > 2\gamma_1(\nu)$. The degree of $\tau_1(\nu)$ is $\deg \tilde{\beta}_1 + \deg \gamma_1 + 3(d+1)u(t_2)\deg \beta_1$, which is at most $(d+1)(1+3u(t_2))\deg \beta_1 \leq 4(d+1)u(t_2)\deg \beta_1$ (since $u \geq 1$). By the definition of $\bar{\gamma}_1$, $\deg \bar{\gamma}_1 = \deg \gamma_1 + \alpha \deg \tau_1$, which is bounded by $\deg \beta_1 + \alpha 4(d+1)u(t_2)\deg \beta_1 < 4(\alpha+1)(d+1)u(t_2)\deg \beta_1$. Now, $\deg \nu^{m_1-1} \deg \beta_1(\nu) \geq (t_1+1)\deg \beta_1 - 1 \geq t_1 \deg \beta_1$. (We have $\deg \beta_1 \geq 1$, provided $\deg g_1 \geq 1$.) Since $t_1 = 4(\alpha+1)(d+1)u(t_2)$, we get that $\deg(\nu^{m_1-1}\beta_1(\nu)) > \deg \bar{\gamma}_1(\nu)$, and we can also obtain $\nu^{m_1-1}\beta_1(\nu) > \bar{\gamma}_1(\nu)$ for $\nu > M_1$ by choosing M_1 somewhat larger if necessary.

Now we repeat all the arguments by defining γ_2 in terms of β_2 (and of ρ and α_{k_2}). Then we define τ_2 in terms of $\tilde{\beta}_2$, γ_2, β_2 and $u(t_3)$. We define $\bar{\gamma}_2$ in terms of γ_2 and τ_2, and ε_2 in terms of γ_2, $\tilde{\beta}_2$ and τ_2. Then we define $g_3(\nu) = \beta_2(\nu)(\gamma_2(\nu)+2)$, and we use 5.4 once more and get defined β_3, $\tilde{\beta}_3$ and m_3. Now reasoning exactly as above we get 5.5.5, 5.5.17, 5.5.18, 5.5.31, 5.5.32, 5.5.36, 55.38 and 5.5.42 to hold when $j = 2$ provided $\nu > M_2$ (where we choose $M_2 > M_1$) for a suitble constant M_3. Then we can proceed and repeat all the arguments to get all the inequalities to hold for $j = 3,4,\ldots,s-1 < p-1$. At each approximation step we only have to claim that $\nu > M_j$ for a certain constant M_j to get all the inequalities to hold. It is therefore clair that we can find one common constant M which can be used in each step.

All the claims on the degree, upper and lower bounds for the coefficients of our final polynomial $\nu^m\beta(\nu)$, $\beta(\nu)$, $\tilde{\beta}(\nu)$, $\gamma(\nu)$,

$\bar{\gamma}(\nu)$ and $\varepsilon(\nu)$ follows from our inductive definitions of t_i, g_i, our repeatedly use of 5.4 and the corresponding properties of the polynomials in 5.4, and also from the definitions of $\gamma(\nu)$, $\bar{\gamma}(\nu)$ and $\varepsilon(\nu)$ in terms of the polynomials we get from 5.4.

Also claims on $u(t)$, M and U follow from corresponding claims in 5.4. This completes the proof of Lemma 5.5.

§6. Proof of Proposition 4.2.

In this section we will prove Proposition 4.2. We will thus have completed the proof of Theorem 4.3. First we will prove a lemma which will give us the existence of "good points" satisfying all the claims of 4.2 apart from condition (6) of 4.2.

Lemma 6.1. Suppose that $f:(R^n,0) \to (R^p,0)$ satisfies $L_1(f),\dots,L_{p+1}(f)$. Let m be the integer of Proposition 3.1. Let $h:(R^n,0) \to (R^p,0)$ be such that $j^m h(0) = 0$, and let for each $t \in [0,1]$, $f_t(x) = f(x)+th(x)$. Consider the map-germ $F(x,t) = (f_t(x),t)$ defined at $\{0\}\times[0,1] \subset R^n \times R$. In this situation there exist a representative of F and a set $\Lambda \subset F(\Sigma(F))-\{0\}\times[0,1]$ such that Λ satisfies all the conditions of 4.2 apart from condition (6).

Proof of Lemma 6.1. Let us first choose a representative of F satisfying the conclusion of 3.1. Then each f_t satisfies $L_1(f_t),\dots,L_{p+1}(f_t)$ for all $t \in [0,1]$, with common Lojasiewicz constants $\bar{C}_1,\dots,\bar{C}_{p+1}$ and exponents $\bar{\alpha}_1,\dots,\bar{\alpha}_{p+1}$. It is clear that for each $t \in [0,1]$ we can find a common bound for all the derivatives of f_t up to a certain order. Now let s be a positive integer, and let g be a positive polynomial. Shrinking the domain of definition of each f_t, we may suppose that each representative of f_t (induced from the representative of F) satisfies the conclusion of Lemma 5.5 with s and g in the hypothesis. (Since we denote the parameter in F by t, we use the symbol s instead of t in hypothesis of 5.5.) Since we have common bounds of the derivatives of each f_t and we also have common Lojasiewicz constants and exponents, the representatives of

f_t satisfying 5.5 can also be defined on a common neighbourhood of $0 \in R^n$. Also the M, K, C and C' of 5.5 can be taken to be the same for each $t \in [0,1]$. Let (y,t) be such that $y \neq 0$, and let $y \in f_t(\Sigma(f_t)) - \{0\}$. Use Lemma 5.5 to find a point $y' = y'_{(y,t)} \neq 0$ satisfying the conclusion of 5.5. By (1) of 5.5 we have that

6.1.1 $$\|y-y'\| \leq Kd(x',D_k(f))^{\nu^m \beta(\nu)},$$

where $x' = (x'_1,\ldots,x'_k)$, $\{x'_1,\ldots,x'_k\} = f_t^{-1}(y') \cap \Sigma(f_t)$, β is some polynomial and $m \geq s \deg \beta(\nu)$. Put $\Lambda = \{(y'_{(y,t)},t) \mid y \in f_t(\Sigma(f_t)), y \neq 0\}$. We want to see that this Λ satisfies the conditions in 4.2 apart from condition (6). Condition (1) of 4.2 follows from (1) of 5.5, putting $\lambda_1 = \tilde{\beta}(\nu)$. Further, condition (2) of 4.2 follows from (2) of 5.5. (Put $\lambda_2 = \beta(\nu)$.) Next put $\lambda_3 = \gamma(\nu)$. Then condition (3) of 4.2 follows from (3) of 5.5. (Note that (3) of 5.5 refers to Proposition 2.4 but (3) of 4.2 refers to the Proposition 2.5. However, since the neighbourhoods in 2.4 and 2.5 are of the same type, we can still use the proof of (3) of 5.5 to prove (3) of 4.2.) Now we want to see that (4) of 4.2 holds. In 5.5 we had $\nu^{m-1}\beta(\nu) > \bar{\gamma}(\nu)$. Let us suppose that $d(x',D_k(f)) < 1/(2C'K)$. (Which is satisfied close to 0.) If $\nu > 2$, then we have that $\nu^m \beta(\nu) > \nu\bar{\gamma}(\nu) > \bar{\gamma}(\nu)+1$. (Recall that all the coefficients of $\bar{\gamma}$ are larger than 1 so $\bar{\gamma}(\nu) > 1$.) From 6.1.1 we now get that

6.1.2 $$\|y-y'\| \leq Kd(x',D_k(f))^{\nu^m \beta(\nu)} \leq 1/(2C')d(x',D_k(f))^{\bar{\gamma}(\nu)}.$$

Now (4) of 4.2 follows by putting $\lambda_4 = \bar{\gamma}(\nu)$. Also (5) of 4.2 follows from (5) of 5.5 by putting $\lambda_5 = \varepsilon(\nu)$.

Now since $\deg \beta$, $\deg \tilde{\beta}$, $\deg \gamma$, $\deg \bar{\gamma}$, $\deg \varepsilon \geq \deg g$ and β, $\tilde{\beta}$, γ, $\bar{\gamma}$, ε have all their coefficients larger than 1, it is clear that we can get λ_1, λ_2, λ_3, λ_4 and λ_5 to be larger than some lower bound we put up in advance by choosing ν and $\deg g$ large.

Now we had that deg β, deg $\tilde{\beta}$, deg γ, deg $\bar{\gamma}$, deg $\varepsilon \leq u(s)$deg g, and that the coefficient of β, $\tilde{\beta}'$ γ, $\bar{\gamma}$ and ε had upper bounds depending on g and $\bar{C}_1,\ldots,\bar{C}_{p+1}$, $\bar{\alpha}_1,\ldots,\bar{\alpha}_{p+1}$ and on upper bounds of the derivatives of f_t up to a certain order. Having already chosen s, g and ν, it follows that λ_1, λ_2, λ_3, λ_4 and λ_5 have upper bounds independent of the particular point (y,t). To obtain $r'/2>r''$ and $r/2>r''$, we use that $\gamma(\nu)>2\bar{\gamma}(\nu)$. Then the inequalities will follow from the definitions of r, r' and r'' provided $d(x',D_k(f))$ is small. It is easy to see that C, C' can be chosen such that $C>C'$. This will imply $r<r'$. This completes the proof of Lemma 6.1.

Since the point set Λ constructed in 6.1 does not satisfy condition (6) of 4.2, we have to see that there is a subset $\tilde{\Lambda} \subset \Lambda$ satisfying condition (4) such that condition (6) also is satisfied. (All the other conditions of 4.2 follow directly from the properties of the points in Λ.) We have the following lemma:

<u>Lemma 6.2</u>. Let Λ be the point set of 6.1 associated to the map F. Then there is a subset $\tilde{\Lambda} \subset \Lambda$ which satisfies all the conditions (including condition (6)) of Proposition 4.2.

<u>Proof</u>. The proof is based on a construction used in Lemma 2.1 p. 72 of [8]. For each positive integer m we divide R^{p+1} into closed cubes each with sides of length $1/2^m$. Let us denote the coordinates in R^{p+1} by $y_1,\ldots,y_p,t$. Then our cubes will be defined by the hyperplanes $y_\nu = j_\nu/2^m$, $\nu = 1,\ldots,p$, $t = j/2^m$ where $j_1,\ldots,j_p,j$ are integers. For each m let $\sum_m$ denote the set of such cubes. Let $K = \{0\}\times R \subset R^{p+1}$, and let S_0 denote

the cubes $L\in\Gamma_0$ such that $d(K,L)\geq\sqrt{p+1}$ where $d(K,L)$ is the distance between K and L. Let us for each m define the following set of cubes $S_m\subset\Gamma_m$ by induction: S_m is the set of cubes $L\in\Gamma_m$ such that L is not contained in any of the cubes in $S_0,\dots,S_{m-1}$, and such that $d(K,L)\geq\sqrt{p+1}/2^m$. Put $I=\bigcup_{m=0}^{\infty}S_m$.

From Lemma 2.1 [8] we get that if a cube L in Γ_m meets a cube in the set S_{m-1} then $d(K,L)\geq\sqrt{p+1}/2^m$, so either L is contained in a cube in $S_0,\dots,S_{m-1}$ or $L\in S_m$. The closed cubes in I therefore cover $R^{p+1}-\{0\}\times[0,1]$, and the cubes in S_m meet only cubes in S_{m-1}, S_m and S_{m+1}. Let $L\in S_m$. Let y_L denote the center of L and λ_L the length of the sides of L. Let $\hat{L}$ be the open cube with center y_L, with length of it sides $(3/2)\lambda_L$ and with all sides parallel to the sides in L. Consider the set $\{\hat{L}\mid L\in I\}$. Then it follows from Lemma 2.1 of [8] that the number of cubes of type $\hat{L}$ that can intersect is at most 4^{p+1}.

Let $q>1$ be an integer. For each $L\in S_m$ divide L into closed subcubes whose sides have the length $1/2^{qm}$.

Let C_m denote the set of centers of all such subcubes when L runs through S_m. Consider the set of the open cubes with center the different points in C_m, with length of the sides equal $1/2^{qm-1}$ and with all sides parallel to the sides of the cubes in S_m. Let $\hat{I}$ denote the set of such cubes when m runs through all the integers. Since I is a cover of $R^p\times R-\{0\}\times R$, $\hat{I}$ becomes an open cover of $R^p\times R-\{0\}\times R$.

Let $(y,t)\in C_m$. Then we also have $(y,t)\in L$ for some $L\in S_m$. Let L' be a cube in Γ_{m-1} such that $L\subset L'$. Then we must have that

6.2.1 $$d(K,L')<\sqrt{p+1}/2^{m-1},$$

otherwize either $L' \in S_{m-1}$ or L' is contained in a cube in S_i, $i<m-1$, which contradicts the fact that $L \in S_m$. Since the diameter of L' is $\sqrt{p+1}/2^{m-1}$ and $(y,t) \in L'$, we get that

6.2.2 $$\|y\| = d(K,(y,t)) \leqslant d(K,L') + \sqrt{p+1}/2^{m-1} \leqslant \sqrt{p+1}/2^{m-2}.$$

On the other hand, since $L \in S_m$, we have that $d(K,L) \geqslant (p+1)/2^m$. So we have that

6.2.3 $$\|y\| = d(K,(y,t)) \geqslant d(K,L) \geqslant \sqrt{p+1}/2^m.$$

From 6.2.3 follows that $1/2^{qm-1} \leqslant 2(\frac{\|y\|}{\sqrt{p+1}})^q < 2\|y\|^q$. (Recall that $q>1$ so $2/(\sqrt{p+1})^q \leqslant 1$.)

Now let F and Λ be as in Lemma 6.1, and suppose that $(y',t') \in F(\int(F)) - \{0\} \times [0,1]$ where $t' \in [0,1]$. Since $\hat{I}$ is an open cover of $R^p \times R - \{0\} \times R$, we can find a cube in $\hat{I}$ with center at (y,t) such that (y',t') belongs to this cube. If $\|y'\|$ is sufficiently small it is clear that the center of the cube in $\hat{I}$ belongs to the set C_m for an $m>0$. It follows that

6.2.4 $$\|(y,t)-(y',t')\| \leqslant \sqrt{p+1}/2^{qm-1}.$$

So 6.2.3 and 6.2.4 imply that

6.2.5 $$\|(y,t)-(y',t')\| \leqslant \sqrt{p+1}\,\|y\|^q.$$

From the way the set Λ was constructed in Lemma 6.1 it follows that there exists a point (y^0,t^0) in Λ such that

6.2.6 $$\|(y',t')-(y^0,t^0)\| \leqslant Kd(x^0,D_k(f))^{\nu^m \beta(\nu)}$$

for some polynomial $\beta(\nu)$, some positive integer m and some constant K satisfying certain conditions given in Lemma 5.5. Actually from the proof of 6.1 we had $t^0 = t'$. In 6.2.6 we have

$x^0 \in (R^n)^k$ with components the distinct points in $f_{t^0}^{-1}(y^0) \cap \Sigma(f_{t^0})$. From 6.2.6 it is easy to prove that we have $\|y'\|^2 < \|y^0\|$, and that $\|y^0\|^2 < \|y'\|$ provided y^0, y' are sufficiently close to 0, and $\nu^m \beta(\nu)$ is sufficiently large. (See the proof of 4.5.6 and 4.5.7 in Sublemma 4.5 for details.) From 6.2.5 it is also easy to see that $\|y'\| > \|y\|^2$ if q is large. Using 4.4. and 5.5 we have that $d(x^0, D_k(f)) \geq \min_i \|x_i^0\|^{\tilde{\beta}(\nu)} \geq \|y^0\|^{2\tilde{\beta}(\nu)}$. Let us suppose that $q > 8\nu^m \tilde{\beta}(\nu)\beta(\nu)$. Then 6.2.5, 6.2.6 and the inequalities above will imply that

6.2.7 $$\|(y,t)-(y^0,t^0)\| \leq (K+\sqrt{p+1})\, d(x^0, D_k(f))^{\nu^m \beta(\nu)}.$$

Now from the proof of Lemma 5.5 follows that $\deg(\nu^m \tilde{\beta}(\nu)\beta(\nu)) \leq (d+1)u(s)\deg g$ (where d and u are defined in 5.5, and we used ν, s and g in the hypothesis of 5.5 in the proof of 6.1). Also bounds of the coefficients of $\tilde{\beta}$ and β are only dependent of $L_1(f_{t^0}), \ldots, L_{p+1}(f_{t^0})$, f_{t^0} and the polynomial g of 5.5. In Lemma 6.1 ν, g and s were chosen independent of t^0, and the Lojasiewicz constants and exponents of f_{t^0} and bounds of the derivatives of f_{t^0} up to a certain order were independent of t^0. Therefore $\nu^m \tilde{\beta}(\nu)\beta(\nu)$ has upper bounds independent of (y^0, t^0). This implies that we can find a common q such that $q > 8\nu^m \tilde{\beta}(\nu)\beta(\nu)$ always holds independent of the particular (y^0, t^0) we consider. Let us suppose that $d(x^0, D_k(f)) < 1/(K+\sqrt{p+1})$. (This is satisfied if y, and therefore y^0, are close to 0.) Then 6.2.7 implies that

6.2.8 $$\|(y,t)-(y^0,t^0)\| \leq d(x^0, D_k(f))^{\nu^m \beta(\nu)-1}.$$

From 5.5 we had that $\nu^{m-1}\beta(\nu) \geq \bar{\gamma}(\nu)$. Since all the coefficients of $\bar{\gamma}$ are ≥ 1, $\nu > 3$ will imply that $\nu^m \beta(\nu) > 3\bar{\gamma}(\nu) \geq \bar{\gamma}(\nu)+2$. It is clear that in the proof of 6.1 we can arrange ourselves such that $\nu > 3$. If the F of 6.1 is defined in a sufficiently small neighbourhood of $\{0\} \times [0,1]$, then we will also obtain $d(x^0, D_k(f)) \leq 1/(4C')$ where C' is the constant appearing in 5.5 and 6.1. 6.2.8 thus implies that

$$6.2.9 \qquad \|(y,t)-(y^0,t^0)\| \leq (1/4C')d(x^0,D_k(f))^{\bar{\gamma}(\nu)} = r''/4.$$

(Recall that r'' in 4.2 is defined in terms of C' and λ_4, and that $\lambda_4 = \bar{\gamma}(\nu)$ by the proof of 6.1.)

Arguing as above, we get for each $(y',t') \in F(\Sigma(F)) - \{0\} \times [0,1]$, $t' \in [0,1]$, associated a point (y,t), which is the center a cube in $\hat{I}$. We also get associated another point $(y^0,t^0) \in \Lambda$ such that 6.2.4-6.2.9 hold. Especially (y',t') belongs to the cube with center (y,t). Collecting all such cubes together, we get a subset $\tilde{I}$ of $\hat{I}$ which covers $F(\Sigma(F)) - \{0\} \times [0,1]$ (where we only consider the points (y',t') with $t' \in [0,1]$). It follows from 6.2.9 that each center (y,t) also is contained in the ball at (y^0,t^0) with radius $r''/4$. From 6.2.3 follows that the diameter of each cube in $\tilde{I}$ with center (y,t) is less than $2\sqrt{p+1}\,\|y\|^q$. Therefore $r''/4 > \sqrt{p+1}\,\|y\|^q$ will be a sufficient condition for the whole cube to be contained in the ball at (y^0,t^0) with radius $r''/2$. From 6.2.9 it is easy to deduce that $\|y\|^2 \leq \|y^0\|$. Now we have that $d(x^0,D_k(f)) \geq \|y^0\|^{2\tilde{\beta}(\nu)}$, and we can suppose that $\|y\| < 1/(4C'\sqrt{p+1})$. Using the definition of r'' (see 6.2.9), $q > 4\tilde{\beta}(\nu)\bar{\gamma}(\nu)+1$ will be a sufficient condition for $r''/4 > \sqrt{p+1}\,\|y\|^q$. Using the upper bound of the degree and the coefficients of $\tilde{\beta}$ and $\bar{\gamma}$, we can also find a q such that $q > 4\tilde{\beta}(\nu)\bar{\gamma}(\nu)+1$ always holds independent of the point (y^0,t^0) we consider. We may therefore suppose that each cube in $\tilde{I}$ with center in (y,t) is contained in the ball with radius

$r''/2$ and center (y^0,t^0). Now for each such cube in $\tilde{I}$ we pick one such point (y^0,t^0) in Λ. Let $\tilde{\Lambda}$ be the subset of Λ we get defined collecting all such points (y^0,t^0) together. (Note that since many points (y',t') belong to the same cube in $\tilde{I}$, (y^0,t^0) is not uniquely determined. It is however clear that we for each cube in $\tilde{I}$ can pick one such point (y^0,t^0).)

Now we want to see that this set $\tilde{\Lambda}$ satisfies our Lemma. First let $(y',t')\in F(\Sigma(F))-\{0\}\times[0,1]$, $t'\in[0,1]$. Now (y',t') is contained in a cube in $\tilde{I}$ with center (y,t), and the whole cube is contained in a ball at a point (y^0,t^0) with radius $r''/2$. Therefore the balls with center $(y^0,t^0)\in\tilde{\Lambda}$ and radius $r''/2$ will cover $F(\Sigma(F))-\{0\}\times[0,1]$. (4) of 4.2 is therefore satisfied.

To see that (6) of 4.2 holds, we consider $\{U_{(y^0,t^0)}(r')\,|\,(y^0,t^0)\in\tilde{\Lambda}\}$, and we have to count the number of balls that can intersect. If we, to each center (y,t) of cubes in $\tilde{I}$, associate the point $(y^0,t^0)\in\tilde{\Lambda}$, we get defined a surjective map $\tilde{I}\to\tilde{\Lambda}$.

Let $\tilde{L}$ be a cube in $\tilde{I}$, and let (y^0,t^0) be the point associated in $\tilde{\Lambda}$ from the above map. Let (y,t) be the center of $\tilde{L}$. (y,t) is also the center of a cube L' such that the length of the side of L' is $1/2^{qm}$ for some m, and L' is a subcube of a cube $L\in S_m$. Let the center of L be (y_L,t_L), and let $\lambda_L = 1/2^m$ be the length of the sides of L. Consider the cube $\hat{L}$ parallel to L with center (y_L,t_L) and with length of it sides equal $(3/2)\lambda_L$. Let us consider the ball $U_{(y^0,t^0)}(r')$. We want to find conditions which will imply that $U_{(y^0,t^0)}(r')\subset\hat{L}$. Since (y,t) belongs to L, and $(r''/4)<r'$, it will follow from 6.2.9 that $2r'<(1/4)\lambda_L$ will be a sufficient condition for $U_{(y^0,t^0)}(r')\subset\hat{L}$. To obtain this recall from the proof of 6.1 that $r' = (1/C')d(x^0,D_k(f))^{\gamma(\nu)}$. Let $x^0 = (x_1^0,\dots,x_k^0)$. Since

f_{t^0} is ∞-$\mathcal{K}$ determined, we have from Sublemma 4.4 that $\|y^0\| \geq \|x_i^0\|^\lambda$ for a $\lambda > 0$. (This λ will be independent of t^0.) Since $d(x^0, D_k(f)) \leq \min_i \|x_i^0\|$, we get that $d(x^0, D_k(f)) \leq \|y^0\|^{1/\lambda}$. Since we can suppose that $C' > 1$, we get that $r' < \|y^0\|^{\gamma(\nu)/\lambda}$. Now $\lambda_L = 1/2^m$, and by 6.2.2 $\|y\| \leq \sqrt{p+1}/2^{m-2}$. Using 6.2.7, we can deduce that with y, y^0 sufficiently close to 0, we have that $\|y\| > \|y^0\|^2$. So we get that $2r' < 2\|y^0\|^{\gamma(\nu)/\lambda} \leq 2\|y\|^{\gamma(\nu)/2\lambda} \leq 2(\sqrt{p+1}/2^{m-2})^{\gamma(\nu)/2\lambda} = 2(4\sqrt{p+1}\lambda_L)^{\gamma(\nu)/2\lambda}$. If we are sufficiently close to 0, we can suppose that $\lambda_L < 1/(16(p+1))$. So if $\gamma(\nu)/2\lambda > 4$, then we get that $(4\sqrt{p+1}\lambda_L)^{\gamma(\nu)/2\lambda} \leq \lambda_L{}^{\gamma(\nu)/4\lambda} < \lambda_L^2$. If we, in the proof of 6.1, suppose that the ν is sufficiently large, we will always have (independent of (y^0, t^0)) that $\gamma(\nu)/2\lambda > 4$. We may therefore suppose that $2r' < 2\lambda_L^2$. If we are close to $\{0\} \times [0,1]$, $\lambda_L < 1/8$, and we will get that $2r' < (1/4)\lambda_L$. We therefore get that $U_{(y^0,t^0)}(r') \subset \hat{L}$.

Now we are prepared to estimate the number of balls $U_{(y^0,t^0)}(r')$, $(y^0,t^0) \in \tilde{\Lambda}$, that can intersect in a common point. We had a surjective map $\tilde{I} \to \tilde{\Lambda}$. Let $\rho: \tilde{\Lambda} \to \tilde{I}$ be a right inverse to this map. Let $(y^0,t^0) \in \tilde{\Lambda}$, and let $\tilde{L} = \rho(y^0,t^0)$. Then we can associate the cubes L', L, $\hat{L}$ such that $L' \subset \tilde{L}$, $L' \subset L \in S_m$, $L \subset \hat{L}$, and $U_{(y^0,t^0)}(r') \subset \hat{L}$. Let (y_1^0, t_1^0), (y_2^0, t_2^0) be two points in $\tilde{\Lambda}$, and let $L_1 \subset \hat{L}_1$, $L_2 \subset \hat{L}_2$ be the cubes we get associated using ρ, such that $L_1 \in S_{m_1}$ and $L_2 \in S_{m_2}$. Let us suppose that $U_{(y_1^0,t_1^0)}(r') \cap U_{(y_2^0,t_2^0)}(r') \neq \emptyset$. Since $U_{(y_i^0,t_i^0)}(r') \subset \hat{L}_i$, $i = 1,2$, we have that $\hat{L}_1 \cap \hat{L}_2 \neq \emptyset$. From the definitions of S_m and $\hat{L}$, it is easy to prove that this implies that $|m_1 - m_2| \leq 2$ if we are sufficiently close to $0 \times R$. Let $\bar{\Lambda} \subset \tilde{\Lambda}$ be a set such that the corresponding balls $\{U_{(y^0,t^0)}(r') \mid (y^0,t^0) \in \bar{\Lambda}\}$ intersect in a common point. For each $(y^0,t^0) \in \bar{\Lambda}$ associate the cubes $\rho(y^0,t^0) = \tilde{L}$, L',

L and $\hat{L}$. By construction ρ is injective. From our construction it is clear that the map defined by $(y^0,t^0) \to L'$ also is injective. Let (y^0,t^0) run through $\bar{\Lambda}$, and consider the set consisting of the cubes L' we get defined. On this set we can define an equivalence relation given by $L_1' \sim L_2'$ iff L_1' and L_2' both are subcubes of the same $L \in S_m$. Since each subcube of L has sides of length $1/2^{mq}$, the toal number of such subcubes is $2^{m(p+1)(q-1)}$. The cardinality of the cubes in each equivalence class is therefore bounded by $2^{m(p+1)(q-1)}$. Also for each such equivalence class we can associate a cube $L \in S_m$, and the cube $\hat{L}$ with the same center as L and whose sides have the length $(3/2)(1/2)^m$. Since each such $\hat{L}$ contains at least one ball of the type $U_{(y^0,t^0)}(r)$, and the number of cubes of type $\hat{L}$ that can intersect is bounded by 4^{p+1}, the number of equivalence classes is also bounded by 4^{p+1}. Let $(y^0,t^0) \in \bar{\Lambda}$, and associate $\tilde{L}$, L', L and $\hat{L}$. Suppose that $L \in S_m$ for an integer m. Now, all the other cubes of type $\hat{L}$ associated to other points in $\bar{\Lambda}$ intersect (since each contains a ball $U_{(y^0,t^0)}(r), (y^0,t^0) \in \bar{\Lambda}$). It will therefore follow from comments already given in this proof that the other cubes of type L associated to other points in $\bar{\Lambda}$ are elements in S_{m-2}, S_{m-1}, S_m, S_{m+1} and S_{m+2}. The number elements in each equivalence class is therefore bounded by $2^{(m+2)(p+1)(q-1)}$, and since ρ is injective, the cardinality of $\bar{\Lambda}$ is bounded by $2^{(m+2)(p+1)(q-1)}4^{p+1}$. If (y,t) is the center of L' (or $\tilde{L}$), we deduce again from 6.2.7 that $\|y\| > \|y^0\|^2$. From 6.2.2 follows that $\|y^0\| < (\sqrt{p+1}/2^{m-2})^{\frac{1}{2}}$. If we are sufficiently close to $0 \times R$, m have to be large so, we can suppose that $\frac{1}{2}^{(m-2)/2} < 1/\sqrt{p+1}$, and we deduce that $\|y^0\| < \frac{1}{2}^{(m-2)/4}$. This implies that

6.2.10 $$2^{(m+2)(p+1)(q-1)}4^{p+1} = 2^{(p+1)[(m+2)(q-1)+2]} \leq \|y^0\|^{-4(p+1)[(m+2)(q-1)+2]/(m-2)}.$$

Now, if (y^0,t^0) is sufficiently close to $\{0\}\times[0,1]$, we will have that $m\geq 3$, and in this case we will get that $[(m+2)(q-1)+2]/(m-2)\leq 2(q-1)$ (provided $q\geq 7$). So the number of balls that can intersect is bounded by $\|y^0\|^{-8(p+1)(q-1)}$ (provided $q\geq 7$). Now if $\{x_1^0,\ldots,x_k^0\} = f_{t^0}^{-1}(y^0)\cap\Sigma(f_{t^0})$, we again have from Sublemma 4.4 that $\|x_i^0\|^{\lambda}<\|y^0\|$ for a suitable λ (which is independent of the particular t^0). We therefore get that $d(x^0,D_k(f))\leq\|x_i^0\|<\|y^0\|^{1/\lambda}$, and the number of intersecting balls is consequently bounded by $d(x^0,D_k(f))^{-8\lambda(p+1)(q-1)}$. We can therefore choose $\lambda_6 = 8\lambda(p+1)(q-1)$ to get (6) of 4.2 to be satisfied. Therefore the subset $\tilde{\Lambda}$ of Λ will satisfy all the conditions in 4.2 (including (6)). This completes the proof of Lemma 6.2.

Now Proposition 4.2 follows directly from Lemma 6.2. With this included in the part of the proof of Theorem 4.3 we already have given in §4, the proof of Theorem 4.3 is complete.

§7. Proof of (e)<=>(t)

Proof of (e)=>(t). This follows almost immediately from the proof of Theorem 4.3. In 4.3 we proved that if f satisfies $L_1(f),\dots,L_{p+1}(f)$, then we can solve 4.3.1 outside 0 such that if $j^m h(0) = 0$ for $m = m(k)$ sufficiently large, then the derivatives of the ξ_t, η_t of 4.3.1 up to order k are $o(\|x\|^{k+1})$, $o(\|y\|^{k+1})$ respectively. If we suppose $j^\infty h(0) = 0$, then we get from this proof that ξ_t, η_t are $o(\|x\|^k)$, $o(\|y\|^k)$, for all k. Let us suppose this and restrict 4.3.1 to the level $t = 0$. Then we get

$$\tau_0 = tf(\xi_0) + \omega f(\eta_0)$$

where

$$\frac{\partial^\alpha}{\partial x^{|\alpha|}}\,\xi_0(x) = o(\|x\|^k), \quad \frac{\partial^\beta}{\partial y^{|\beta|}}\,\eta_0(y) = o(\|y\|^k)$$

for any k, and any pair of multiindices α, β. Putting $\xi_0(0) = 0$, $\eta_0(0) = 0$, it follows that ξ and η become C^∞ as germs on R^n, R^p respectively. Now we had that

$\tau_0(x) = \sum_{j=1}^{n} h_j(x)\frac{\partial}{\partial y_j}$ of where each h_j was an arbitrarily germ with $j^\infty h_j(0) = 0$. So $\tau_0(x)$ can be taken to be any element of $m_n^\infty V(f)_0$. From above we get that

$$\tau_0 = tf(\xi_0) + \omega f(\eta_0)$$

on R^n with $\xi_0 \in V(R^n)_0$, $\eta_0 \in V(R^p)_0$. From this we get that (e)=>(t). In fact since we get that $j^\infty\xi_0(0) = 0$ and that $j^\infty\eta_0(0) = 0$, we have proved the implication

$$(e) \Rightarrow m_n^\infty V(f)_0 \subseteq tf(m_n^\infty V(R^n)_0) + \omega f(m_p^\infty V(R^p)_0).$$

Let us now consider the other implication.

Proof of (t)=>(e). Suppose (e) fails. Then we will construct a vectorfield $\tau\in m_n^\infty V(f)_0$ such that $\tau\notin T(f)$. Suppose that $L_\ell(f)$ fails for an ℓ, $1\leq\ell\leq p+1$, but each $L_j(f)$ is satisfied for any $1\leq j<\ell$.

For each $x = (x_1,\ldots,x_\ell)\in(R^n)^\ell$ we consider the map $\hat{a}_f^\ell(x) = \sum_{1\leq i,j\leq\ell}\|f(x_i)f(x_j)\|^2+\det(A_f^\ell(x)(A_f^\ell(x))^t)$, and since $L_\ell(f)$ fails, we can pick a sequence $x^k = (x_1^k,\ldots,x_\ell^k)\in(R^n)^\ell$ such that $x^k \to 0\in(R^n)^\ell$, and such that $\hat{a}_f^\ell(x^k)$ is flat with respect to $d(x^k,D_\ell(f))$. (We say that $\hat{a}_f^\ell(x^k)$ is flat with respect to $d(x^k,D_\ell(f))$ iff $\hat{a}_f^\ell(x^k) = o(d(x^k,D_\ell(f))^m)$ for each integer m.) First we wish to prove that $\hat{a}_f^\ell(x^k)$ in fact is flat with respect to any of $\|x_i^k-x_j^k\|$, $\|x_i^k\|$, $1\leq i,j\leq\ell$, $i\neq j$.

For any i, $\|x_i^k\|\geq d(x^k,D_\ell(f))$, so $\hat{a}_f^\ell(x^k)$ is flat with respect to $\|x_i^k\|$, $i = 1,\ldots,\ell$. So if $\ell = 1$, then there is nothing more to prove. Suppose that $\ell>1$. Then $L_1(f)$ is satisfied. From Lemma 3.10 and Remark 3.11 we get that $J(f)(x)\geq d(x,\Sigma(f))^{\alpha'}$ for a suitable $\alpha'>0$. Now suppose there exists a $\beta>0$ such that for one i we have that $J(f)(x_i^k)\geq d(x^k,D_\ell(f))^\beta$ for all sufficiently large k. Without loss of generality let us suppose that $i = \ell$. Put $\hat{x}^k = (x_1^k,\ldots,x_{\ell-1}^k)$. Since $L_{\ell-1}(f)$ is satisfied, we have that $\hat{a}_f^{\ell-1}(\hat{x}^k)\geq C_{\ell-1}d(\hat{x}^k,D_{\ell-1}(f))^{\alpha_{\ell-1}}$ for suitble $C_{\ell-1}$, $\alpha_{\ell-1}$. Now, after a slight modification of the proof of 3.5, we will obtain, using $J(f)(x_i^k)\geq d(x^k,D_\ell(f))^\beta$ instead of $J(f)(x_i^k)\geq\|x_i^k\|^\beta$, that $\hat{a}_f^\ell(x^k)\geq C_\ell d(x^k,D_\ell(f))^{\alpha_\ell}$ for suitable constants $C_\ell,\alpha_\ell>0$. This contradicts the fact that $\hat{a}_f^\ell(x^k)$ is flat with respect to $d(x^k,D_\ell(f))$. It follows that $J(f)(x_i^k)$ has to be flat with respect to $d(x^k,D_\ell(f))$ for any i.

Consider two indices i, j, and let $\bar{x}_i^k\in\Sigma(f)$ be such that

$\|x_i^k-\bar{x}_i^k\| = d(\bar{x}_i^k,\Gamma(f))$. (If k is large such an $\bar{x}_i^k$ always exists.) It follows from above that $J(f)(x_i^k)\geq\|x_i^k-\bar{x}_i^k\|^{\alpha'}$. So from above we get that $\|x_i^k-\bar{x}_i^k\|$ is flat with respect to $d(x^k,D_\ell(f))$. Now we have that $\|x_j^k-\bar{x}_i^j\|+\|\bar{x}_i^k-x_i^k\|\geq d((x_i^k,x_j^k),\Delta\Gamma(f))\geq d(x^k,D_\ell(f))$. Since we have deduced that $\|x_i^k-\bar{x}_i^k\|$ is flat with respect to $d(x^k,D_\ell(f)$, we also deduce that for large k we have $\|x_j^k-\bar{x}_i^k\|\geq d(x^k,D_\ell(f))^2$. So we get that $\|x_j^k-x_i^k\|\geq\|x_j^k-\bar{x}_i^k\|-\|\bar{x}_i^k-x_i^k\|\geq d(x^k,D_\ell(f))^2-\|\bar{x}_i^k-x_i^k\|$. Again since $\|\bar{x}_i^k-x_i^k\|$ is flat with respect to $d(x^k,D_\ell(f))$, we can easily deduce from the above inequality that for large k we have that $\|x_j^k-x_i^k\|\geq d(x^k,D_\ell(f))^3$. The above argument is valid for any i,j, and from this follows that $\hat{a}_f^\ell(x)$ is flat with respect to $\|x_j^k-x_i^k\|$, and $\|x_i^k\|$ for any i,j, $i\neq j$.

Now let us for each n consider the non-negative definit matrix.

$$A_k = A_f^\ell(x^k)A_f^\ell(x^k)^t(R^p\times J^p(n,p))^\ell \to (R^p\times J^p(n,p))^\ell.$$

We will consider the two cases:

I $\quad A_f^\ell(x^k)$ is surjective for large n.

II $\quad$ There is a subsequence $\{x^{k_s}\}$ of $\{x^k\}$ such that the $A_f^\ell(x^{k_s})$ is not surjective for each s.

In both cases we will construct $\tau\in m_n^\infty V(f)_0$ such that $\tau\notin T(f)$.

Let us first consider case I. Then A_k has a base of eigenvectors with positive eigenvalues. Let λ_k be the smallest eigenvalue, and $v_k\in(R^p\times J^p(n,p))^\ell$ an eigenvector corresponding to this eigenvalue. Since $\hat{a}_f^\ell(x^k)$ is flat with respect to $\|x_i^k-x_j^k\|$ and $\|x_i^k\|$ for all i,j, we may (by choosing a subsequence of x^k if necessary) suppose that

$$\hat{a}_f^\ell(x^k)<\Big(\Big(\prod_{\substack{1\leq i,j\leq\ell\\ i\neq j}}\|x_i^k-x_j^k\|\Big)\Big(\prod_{1\leq i\leq\ell}\|x_i^k\|\Big)\Big)^{2(k+1)} \quad \text{for all } k.$$

Now let us scale the vector v_k such that we suppose that the norm of v_k is exactly $\left(\left(\prod_{\substack{1\leq i,j\leq \ell\\ i\neq j}} \|x_i^k - x_j^k\|\right)\left(\prod_{1\leq i\leq \ell} \|x_i^k\|\right)\right)^k$. Then $\|v^k\|$ is flat with respect $\|x_i^k - x_j^k\|$ and $\|x_i^k\|$ for all i, j. Using Lemma 2.2 of [13], we can now construct a vectorfield $\tau \in V(f)_0$ such that $j^\infty \tau(0) = 0$, and such that $j^p\tau(x^k) = v_k$ holds for a subsequence of $\{x^k\}$.

Now we want to show that $\tau \notin T\mathcal{A}(f)$. Let us suppose that $\tau = tf(\xi) + \omega f(\eta)$, we want to obtain a contradiction.

For each x_i^k we must have

7.1 $$j^p\tau(x_i^k) = A_f^1(x_i^k)(j^p\xi(x_i^k), j^p\eta(f(x_i^k)).$$

Put

7.2 $$u_i^k = A_f^1(x_i^k)(0, j^p\eta(f(x_i^k)) - j^p\eta(f(x_\ell^k))$$

Then we get that

7.3 $$(j^p\tau(x_1^k), \ldots, j^p\tau(x_\ell^k)) - (u_1^k, \ldots, u_{\ell-1}^k, 0)$$

$$= A_f^\ell(x^k)(j^p\xi(x_1^k), \ldots, j^p\xi(x_\ell^k), j^p\eta(f(x_\ell^k)).$$

Now consider the matrix $C_k = (A_f^\ell(x^k))^t A_k^{-1}$. ($A_k$ is invertible since $A_f^\ell(x^k)$ is surjective.) Then C_k is the right inverse to $A_f^\ell(x^k)$ such that for all vectors v, $C_k(v)$ is the vector in $A_f^\ell(x^k)^{-1}(v)$ with the smallest norm. Put $\bar{u}^k = C_k(u_1^k, \ldots, u_{\ell-1}^k, 0)$. Then we get that $A_f^\ell(x^k)((j^p\xi(x_1^k), \ldots, j^p\xi(x_\ell^k), j^p\eta(f(x_\ell^k)) + \bar{u}^k) = (j^p\tau(x_1^k), \ldots, j^p(x_\ell^k))$. So we have that

7.4 $$\|(j^p\xi(x_1^k), \ldots, j^p\xi(x_\ell^k), j^p\eta(f(x_\ell^k))) + \bar{u}^k\|$$

$$\geq \|C_k(j^p\tau(x_1^k), \ldots, j^p\tau(x_\ell^k))\|.$$

Expresing $(u_1^k, \ldots, u_{\ell-1}^k, 0) \in (R^p \times J^p(n,p))^\ell$ as an ortogonal sum of eigenvectors for A_k, it is easy to see that

7.5 $\quad \|\bar{u}^k\| = \|C_k(u_a^k,\dots,u_{\ell-1}^k,0)\| \leq (t/\lambda_k^{\frac{1}{2}})\|(u_1^k,\dots,u_{\ell-1}^k,0)\|$

where $t = \dim_R(R^p \times J^p(n,p))^\ell$.

Now we have $u_i^k = A_f^{\frac{1}{2}}(x_i^k)(0, j^p\eta(f(x_i^k)) - j^p\eta(f(x_\ell^k)))$, and since the derivatives of f and of η are bounded, we get that $\|u_i^k\| \leq K\|f(x_i^k) - f(x_\ell^k)\|$ for some $K>0$, where K bounds the derivatives of f and η up to order $p+1$. Also on a subsequence of x^k, we have that $(j^p\tau(x_1^k),\dots,j^p\tau(x_\ell^k)) = v_k$, so we have that $\|C_k(j^p\tau(x_1^k),\dots,j^p\tau(x_\ell^k))\| = \|v_k\|/\lambda_k^{\frac{1}{2}}$ on a subsequence of $\{x^k\}$. Using this and 7.5 in 7.4, we get (after a redefinition of K) that

7.6
$$\|(j^p\xi(x_1^k),\dots,j^p\xi(x_\ell^k), j^p\eta(f(x_\ell^k)))\|$$
$$\geq (1/\lambda_k^{\frac{1}{2}})(\|v_k\| - Kt\sum_{i,j}\|f(x_i^k) - f(x_j^k)\|)$$

on a subsequence of $\{x^k\}$. Since we have supposed that $\|f(x_i^k) - f(x_j^k)\|^2 \leq \hat{a}_f(x^k) < ((\prod_{\substack{1\leq i,j\leq\ell\\ i\neq j}}\|x_i^k - x_j^k\|)(\prod_{1\leq i\leq\ell}\|x_i^k\|))^{2(k+1)}$, and $\|v_k\| = ((\prod_{\substack{1\leq i,j\leq\ell\\ i\neq j}}\|x_i^k - x_j^k\|)(\prod_{1\leq i\leq\ell}\|x_i^k\|))^k$, we get that

7.7
$$\|(j^p\xi(x_1^k),\dots,j^p\xi(x_\ell^k), j^p\eta(f(x_\ell^k)))\|$$
$$\geq (1/(2\lambda_k^{\frac{1}{2}}))(\prod_{\substack{1\leq i,j\leq\ell\\ i\neq j}}\|x_i^k - x_j^k\|)(\prod_{1\leq i\leq\ell}\|x_i^k\|))^k \quad \text{for } k \text{ large.}$$

Now $\hat{a}_f^\ell(x^k) \geq a_f^\ell(x^k) = \det(A_k) \geq \lambda_k^t$. So $\lambda_k^{\frac{1}{2}}$ also has to be flat with respect to $\|x_i^k - x_j^k\|, \|x_i^k\|$. We could therefore in the beginning of the proof also supposed that

$$2\lambda_k^{\frac{1}{2}} \leq ((\prod_{\substack{1\leq i,j\leq\ell\\ i\neq j}}\|x_i^k - x_j^k\|)(\prod_{1\leq i\leq\ell}\|x_i^k\|))^{k+1}.$$

Let us suppose this, then we get from 7.7 that

7.8 $$\|(j^p\xi(x_1^k),\ldots,j^p\xi(x_\ell^k),j^p\eta(fx_\ell^k))\| \geq ((\prod_{\substack{1\leq i,j\leq \ell\\ i\neq j}}\|x_i^k-x_j^k\|)(\prod_{1\leq i\leq \ell}\|x_i^k\|))^{-1}.$$

It follows that $\|(j^p\xi(x_1^k),\ldots,j^o\xi(x_\ell^k),j^p\eta(f(x_\ell^k))\| \to \infty$ when $k \to \infty$ (since $x_i^k \to 0$). This contradicts the fact that ξ, η are C^∞. So we cannot have $\tau\in T\mathcal{A}(f)$.

Let us now consider case II. We might as well suppose that each $A_f^\ell(x^k)$ is not surjective. For each k, let v_k be a vector in the orthogonal compliment to $\operatorname{im} A_f^\ell(x^k)$ with norm exactly $((\prod_{\substack{1\leq i,j\leq \ell\\ i\neq j}}\|x_i^k-x_j^k\|)(\prod_{1\leq i\leq \ell}\|x_i^k\|))^k$. Again, let $\tau\in V(f)_0$ be such that $j^\infty\tau(0) = 0$, and $(j^p\tau(x_1^k),\ldots,j^p\tau(x_\ell^k)) = v_k$ for infinitely many k. Suppose that $\tau = tf(\xi)+\omega f(\eta)$. Again put $u_i^k = A_f^1(x_i^k)(0,j^p\eta(f(x_i^k))-j^p\eta(f(x_\ell^k)))$. As in 7.3 we get that

7.9 $$(j^p\tau(x_1^k),\ldots,j^p\tau(x_\ell^k))-(u_1^k,\ldots,u_{\ell-1}^k,0) = A_f^\ell(x^k)((j^p\xi(x_1^k),\ldots,j^p\xi(x_\ell^k),j^p\eta(f(x_\ell^k))).$$

So we get that $(j^p\tau(x_1^n)\ldots,j^p\tau(x_\ell^n))-(u_1^k,\ldots,u_{\ell-1}^k,0)$ is in $\operatorname{im} A_f^\ell(x^k)$. Now $(j^p(\tau(x_1^k),\ldots,j^p\tau(x_\ell^k))$ is in $(\operatorname{im}A_f^\ell(x^k))^\perp$ (for infinitely many k), and has norm exactly equal

$(\prod_{\substack{1\leq i,j\leq \ell\\ i\neq j}}\|x_i^k-x_j^k\|)(\prod_{1\leq i\leq \ell}\|x_i^k\|))^k$ for infinitely many k. We therefore get that $\|(u_1^k,\ldots,u_{\ell-1}^k,0)\|\geq((\prod_{\substack{1\leq i,j\leq \ell\\ i\neq j}}\|x_i^k-x_j^k\|)(\prod_{1\leq i\leq \ell}\|x_i^k\|))^k$ for infinitely many k.

On the other, hand if we argue as in the case I, we get that $\|u_i^k\|\leq K\|f(x_i^k)-f(x_\ell^k)\|$. Now again we can suppose that

$$\|f(x_i^k)-f(x_j^k)\|^2 \leqslant \hat{a}_f^\ell(x^k) \leqslant \Big(\Big(\prod_{\substack{1\leqslant i,j\leqslant \ell\\ i\neq j}} \|x_i^k-x_j^k\|\Big)\Big(\prod_{1\leqslant i\leqslant \ell} \|x_i^k\|\Big)\Big)^{2(k+1)},$$ which

gives us

$$\|(u_1^k,\ldots,u_{\ell-1}^k,0)\| \leqslant (\ell-1)K\Big(\prod_{\substack{1\leqslant i,j\leqslant \ell\\ i\neq j}} \|x_i^k-x_j^k\| \prod_{1\leqslant i\leqslant \ell} \|x_i^k\|\Big)^{k+1}$$

which again contradicts to the fact that

$$\|(u_1^k,\ldots,u_{\ell-1}^k,0)\| \geqslant \Big(\Big(\prod_{\substack{1\leqslant i,j\leqslant \ell\\ i\neq j}} \|(x_i^k-x_j^k)\|\Big)\Big(\prod_{1\leqslant i\leqslant \ell} \|x_i^k\|\Big)\Big)^k$$

for infinitely many k. It follows that we cannot have $\tau = tf(\xi)+\omega f(\eta)$ in this case either. This completes the proof of (t)=>(e).

§8. Some examples

These examples are taken from [2]:

1. $f(x,y) = (x^2+y^2, x(x^2+y^2))$
2. $f(x,y) = (x(x^2+y^2), y^2(x^2+y^2))$
3. Let $g:(R^2,0) \to (R^2,0)$ be finitely $\mathcal{A}$-detrmined and analytic, and let $h:(R^2,0) \to (R^2,0)$ be defined by $h(x,y) = (x(x^2+y^2), y(x^2+y^2))$. Put $f = g \circ h$.
4. Let $f:(R^n,0) \to (R^2,0)$ be defined by
$$f(x_1,\ldots,x_n) = (x_1, \sum_{i=1}^{n-2} x_i^2 + (x_i^2 + (x_{n-1}^2 + x_n^2)^2 + x_1^2(x_{n-1}^2 + x_n^2)).$$

In 1, 2, 3, it is easy to see that the ideal generated by the components of f and the 2×2 minors of Df, is contained in the ideal generated by x^2+y^2. In 4 the same ideal is contained in the ideal generated by $x_1,\ldots,x_{n-2}, x_{n-1}^2+x_n^2$. If f is finitely $\mathcal{A}$ determined, then f is also finitely $\mathcal{K}$ determined, so any of these ideals have to contain a power of the maximal ideal m_n. Now, since the above ideals are contained in ideals with fewer generators than the dimension of the source, the ideals cannot contain a power of m_n. So neither of the f's in 1-4 are finitely $\mathcal{K}$ (nor $\mathcal{A}$) determined. On the other hand it is easy to check that f in 1, 2, 4 only have fold-type singularities outside 0, and that the image of critical points do not intersect in the target. So in the case 1, 2, 4, f is multistable outside 0 and analytic. It follows from Remark 0,2 that f satisfies Theorem 0.1. So f is ∞-$\mathcal{A}$ determined and finitely $\mathcal{A}^{(k)}$ determined for every $k<\infty$.

In (3) h is a homeomorphism, and a local diffemorphism outside 0. Since g is supposed to be finitely $\mathcal{A}$-determined and analytic, g_C, and consequently g is multistable outside 0. The composed map f has therefore to be multistable outside 0. Since f also is analytic, it follows from 0.1 and 0.2 that f is ∞-$\mathcal{A}$ determined, and finitely $\mathcal{A}^{(k)}$ determined for every $k<\infty$.

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Appendix A. On the Mather Division Theorem.

In this appendix we want to derive some consequences of the Mather Division Theorem. These consequences are rather wellknown versions of the Preparation Theorem, but we need to supply these versions with explicit estimates of bounds of the derivatives of the dividend and remainder terms.

Let $\Gamma(t,u) = \Gamma_p(t,u)$ be the polynomial on $R \times R^p$ defined by

$$\Gamma(t,u) = t^p + \sum_{i=1}^{p} u_i t^{p-i}.$$

Define $R(t,u) = R_p(t,u)$ by

$$R(t,u) = \sum_{i=1}^{p} u_i t^{p-i}.$$

Let $Z = \{(t,u) | \Gamma(t,u) = 0\}$, and let N be an open neighbourhood of Z which we keep fixed throughout this appendix. For each $u \in R^p$, put $N(u) = \{t \in R | (t,u) \in N\}$. For $X \subseteq R^p$, put $N(X) = \bigcup_{u \in X} N(u)$.
Let $f = (f_1, \ldots, f_m) \in C^\infty(R^n, R^m)$ for some n,m, and let $X \subseteq R^n$ be a set. For each nonnegative integer k define

$$\| f \|_{k,X} = \sup_{\substack{x \in X \\ |\alpha| \leq k \\ 1 \leq i \leq m}} \left| \frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) \right|$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex. With this notation Mather proves the following theorem in [1].

Theorem A.1. (Mather Division Theorem). There exist mappings $Q = Q_p : C^\infty(R) \times R \times R^p \to R$, and $H = H_p : C^\infty(R) \times R^p \to R^p$ having the following properties:

(a) For any $f \in C^{\infty}(R)$, $t \in R$ and $u \in R^p$, we have

(1) $$f(t) = \Gamma_p(t,u)Q_p(f,t,u) + R_p(t,H_p(f,n)).$$

(b) Q and H are R linear in the first variable.

(c) If $f \in C^{\infty}(R)$, $u \in R^p$, and f vanishes on $N(u)$, then $H(f,u) = 0$.

(d) For all $f \in C^{\infty}(R)$, the mappings $Q_f : R \times R^p \to R$ and $H_f : R^p \to R^p$, defined by $Q_f(t,u) = Q(f,t,u)$ and $H_f(u) = H(f,u)$, are C^{∞}.

(e) For any non-negative integer k, there exists a non-negative integer $K = K_k$ such that the following hold:

(i) For any compact subset Y of R^p, there exists $C = C_{k,Y} > 0$ such that, for any $f \in C^{\infty}(R)$,

(2) $$\|H_f\|_{k,Y} \leq C\|f\|_{K,N(Y)}.$$

(ii) Let $\pi : R \times R^p \to R^p$ and $\rho : R \times R^p \to R$ denote the projections. For any compact set Y of $R \times R^p$, there exists $C = C_{k,Y} > 0$ such that, for any $f \in C^{\infty}(R)$,

(3) $$\|Q_f\|_{k,Y} \leq C\|f\|_{K,Y'}$$

where $Y' = N(\pi Y) \cup \rho Y$.

Remark A.2. The norms on the functions we have defined above are somewhat different from the norms defined by Mather [1]. It is however easy to see that our norms are equivalent with those in [1], so the Division Theorem in [1] carries over with the above norms.

From Theorem A.1 we want to derive other familar versions of the Preparation Theorem and also use A.1(e) to get explicit bounds of the derivatives of the dividend and the remainder terms.

First suppose that $f: R \times U \to R$ is a C^∞ function with U some open set in R^n. Then it follows from A.1(a) that

$$f(t,x) = \Gamma_p(t,u) q_f(t,u,x) + R_p(t, h_f(u,x)),$$

with $q_f(t,u,x) = Q(f_x, t, u)$, $h_f(u,x) = H(f_x, u)$, and where f_x is defined by $f_x(t) = f(t,x)$. Then, by the proposition on p. 103 in [1], we have that q_f and h_f are C^∞, and we also have that $\frac{\partial^{|\alpha|}}{\partial x^\alpha} q_f = q_{\frac{\partial^{\alpha}}{\partial x^\alpha} f}$, $\frac{\partial^{|\alpha|}}{\partial x^\alpha} h_f = h_{\frac{\partial^{|\alpha|}}{\partial x^\alpha} f}$, for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$. The next lemma gives us estimates of the derivatives of q_f and h_f.

<u>Lemma A.3</u>. For any non-negative integer k, there exists a non-negative integer $K = K_k$ such that the following hold:

(i) Let $\pi_1: R^p \times U \to R^p$, $\pi_2: R^p \times U \to U$ denote the projections. Let $Y \subseteq R^p \times U$ be any compact set. Then there exists a constant $C = C_{k,Y} > 0$ such that

(1) $$\|h_f\|_{k,Y} \leq C \|f\|_{K, N(\pi_1(Y)) \times \pi_2(Y)}.$$

(ii) Let $\rho_1: R \times R^p \times U \to R$, $\rho_2: R \times R^p \times U \to R \times R^p$ $\rho_3: R \times R^p \times U \to R^p$, $\rho_4: R \times R^p \times U \to U$ denote the projections. If $Y \subseteq R \times R^p \times U$ is any compact set, then there exists a $C = C_{k,Y} > 0$ such that

(2) $$\|q_f\|_{k,Y} \leq C \|f\|_{k,Y'},$$

where $Y' \subseteq R \times U$ denotes the compact set $(N(\rho_3(Y)) \cup \rho_1(Y)) \times \rho_4(Y)$.

Proof of Lemma A.3. Let us first prove the inequality (1). For each multiindex $(\alpha,\beta) = (\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_p)$, consider $\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial u^\beta} h_f = \frac{\partial^{|\beta|}}{\partial u^\beta} h_{\frac{\partial^{|\alpha|} f}{\partial x^\alpha}}$. For a fixed $x \in \pi_2(Y)$, let $Y_x = \pi_1(\pi_2^{-1}(x) \cap Y)$. Since $Y_x \subseteq \pi_1(Y)$, we have from A.1(2):

$$\| H_{(\frac{\partial^{|\alpha|} f}{\partial x^\alpha})_x} \|_{k,Y_x} \leq C_{k,\pi_1(Y)} \| (\frac{\partial^{|\alpha|} f}{\partial x^\alpha})_x \|_{K_k, N(\pi_1(Y))}.$$

Now,

$$\| (\frac{\partial^{|\alpha|} f}{\partial x^\alpha})_x \|_{K_k, N(\pi_1(Y)} \leq \|f\|_{K_k+|\alpha|, N(\pi_1(Y))\times\pi_2(Y)}.$$

So we have

$$\|h_f\|_{k,Y} \leq C_k \|f\|_{K_k+k, N(\pi_1(Y))\times\pi_2(Y)}.$$

Redefining K_k, we get (1).

To prove (2), let $(s,\alpha,\beta) = (s,\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_p)$ be a multiindex, and consider $\frac{\partial^s}{\partial t^s}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial u^\beta} q_f = \frac{\partial^s}{\partial t^s}\frac{\partial^{|\beta|}}{\partial u^\beta} q_{\frac{\partial^{|\alpha|}}{\partial x^a} f}$. For a fixed $x \in \rho_4(Y)$, let $Y_x = \rho_2(\rho_4^{-1}(x) \cap Y)$. Since $Y_x \subseteq \rho_2(Y)$, we have from A.1(3):

$$\| Q_{(\frac{\partial^{|\alpha|} f}{\partial x^\alpha})_x} \|_{k,Y_x} \leq C_{k,\rho_2(Y)} \| (\frac{\partial^{|\alpha|} f}{\partial x^\alpha})_x \|_{K_k, (N(\rho_3(Y)) \cup \rho_1(Y))}.$$

Now,

$$\| (\frac{\partial^{|\alpha|} f}{\partial x^\alpha})_x \|_{K_k, N(\rho_3(Y)) \cup \rho_1(Y)} \leq \|f\|_{K_k+|\alpha|, (N(\rho_3(Y)) \cup \rho_1(Y))\times\rho_4(Y)}.$$

So we get that

$$\|q_f\|_{k,Y} \leq C_{k,\rho_2(Y)} \|f\|_{K_k+k, (N(\rho_3(Y)) \cup \rho_1(Y))\times\rho_4(Y)}.$$

Redefining K_k, we get (2).

Now we have:

<u>Proposition A.4.</u> Let f,g be smooth functions defined on $R^n \times R$, and suppose $f(0,t) = t^p w(t)$, where $w(0) \neq 0$. Then there exist constants $C = C_f > 0$ and $\beta \geq 0$ such that the following hold.

Let $r = (1/C_f)|w(0)|^{\beta}$, and put $B_r^1 = \{x \in R^n \mid \|x\| < r\}$, $B_r^2 = \{(t,x) \in R \times R^n \mid \|(t,x)\| < r\}$. Then there exist a map $\bar{h}_g = (\bar{h}_g^1, \ldots, \bar{h}_g^p) : B_r^1 \to R^p$ and a function $\bar{q}_g : B_r^2 \to R$ such that

$$(1) \qquad g(t,x) = \bar{q}_g(t,x) f(t,x) + \sum_{i=1}^{p} \bar{h}_g^i(x) t^{p-1}.$$

Further, for each non-negative integer k, there exist real positive constants $C_{k,f}$, $C_{k,g}$, and constants $\beta_k \geq 0$ such that

$$\|\bar{h}_g\|_{k,B_r^1} \leq C_{k,f} C_{k,g} |1/w(0)|^{\beta_k}$$
$$\|\bar{q}_g\|_{k,B_r^2} \leq C_{k,f} C_{k,g} |1/w(0)|^{\beta_k}.$$

If $|w(0)| \geq 1$, then we may take $\beta = \beta_k = 0$. Otherwise, the β, β_k are independent of f and only dependent of n,p. $C_f, C_{k,f}, C_{k,g}$ are dependent of f,g in the following way. There exist positive polynomials P, P_k, non-negative integers m, m_k and a compact set $M \subseteq R \times R^n$, which all only depend on n,p and not on f,g, such that $C_f, C_{k,f}, C_{k,g}$ can be defined by:

$$C_f = P(\|f\|_{m,M}), \quad C_{k,f} = P_k(\|f\|_{m_k,M}), \quad C_{k,g} = P_k(\|g\|_{m_k,M}).$$

<u>Proof</u>. (1) is a well known version of the Preparation Theorem with the folowing standard proof: (See [3], p. 93). Use (1) of A.1 to write

$$f(t,x) = \Gamma_p(t,u)q_f(t,u,x) + \sum_{i=1}^{p} h_f^i(u,x)t^{p-i},$$

$$g(t,x) = \Gamma_p(t,u)q_g(t,u,x) + \sum_{i=1}^{p} h_g^i(u,x)t^{p-i}.$$

Define $\phi: R^p \times R^n \to R^p \times R^n$ by $\phi(u,x) = (h_f(u,x),x) = (h_f^1(u,x),\ldots,h_f^p(u,x),x)$. Then $\det D\phi(0) = (-1)^p(q_f(0))^p = (-1)^p w(0)^p \neq 0$. It follows that ϕ is invertible in a neighbourhood of 0. Let us denote this inverse by $(u,x)\to(\theta(u,x),x)$. Thus, in a neighbourhood of 0, we can write

$$g(t,x) = f(t,x)\bar{q}_g(t,x) + \sum_{i=1}^{p} \bar{h}_g^i(x)t^{p-1},$$

with $\bar{q}_g(t,x) = \dfrac{q_g(t,\theta(0,x),x)}{q_f(t,\theta(0,x),x)}$, and $\bar{h}_g^i(x) = h_g^i(\theta(0,x),x)$.

From the construction above, we see that if θ is defined on a ball of radius r, and $q_f \neq 0$ on a corresponding ball with radius r, then also $\bar{q}_g$, $\bar{h}_g$ are defined on balls with radius r. Our next task is to estimate this r, and also etimate bounds of the derivatives of $\bar{q}_g$, $\bar{h}_g^i$ constructed in the above manner. Using the estimates of A.3, and calculating the derivatives by the chain rule, it is in principle straitforward to do this estimation. To write this out in every detail is however rather painful, so we prefer to be sketchy. Suppose first that $|w(0)|<1$.

Consider $\lambda(u,x) = (u,x)-(D\phi(0))^{-1}\circ\phi$ (λ exists since $\det D\phi(0)\neq 0$). We have $D\lambda(0) = 0$, so by continuity, there exists $r>0$ such that $\|D\lambda(u,x)\|<\frac{1}{2}$ for $\|(x,u)\|<r$. (In this special case $\|\ \|$ denotes the L_2 norm of $D\lambda(x,u)$.) By the proof of the inverse function theorem (see [4], p.12-13), ϕ^{-1} exists on $B_{r/2} = \{(u,x)\,|\,\|(u,x)\|<\frac{r}{2}\}$, and ϕ^{-1} takes values in $B_r = \{(u,x)\,|\,\|(u,x)\|<r\}$. We need to estimate this r.

Using Taylors-formula of order 2, we see that $\|D\lambda(x,u)\|<\frac{1}{2}$ on a ball of radius $1/\bar{K}$, where $\bar{K}$ is a bound derived from bounds of the second order derivatives of λ. Now bounds for the derivatives of λ can be estimated from the derivatives of ϕ. Supposing ϕ is defined on a ball with radius 1, we can by A.3 estimate bounds of the derivatives of ϕ by the derivatives of f restricted to a sufficiently large compact $\bar{M}$. Now calculating the derivatives of λ, and recalling that $|\det D\phi(0)| = |w(0)|^p$, we see that the second order derivatives of λ can be bounded by an expression $\bar{C}_f|w(0)|^p = \bar{P}(\|f\|_{\bar{m},\bar{M}})/|w(0)|^p$, where $\bar{P}>1$ is some positive polynomial, $\bar{m}$ is some integer and $\bar{M}$ is a compact set which all are independent of f. Since we may derive a similar bound for $\bar{K}$, we may suppose that ϕ^{-1} and θ are defined on the ball $B_{r/2}$, with $r = |w(0)|^p/\bar{C}_f$ and $\bar{C}_f$ is the expression described above.

Next, we need perhaps to make r smaller to obtain $q_f(t,\theta(0,x),x)\neq 0$ on $B_{r/2}$. In fact, we want to estimate a new r such that $|q_f(t,u,x)|>\frac{1}{2}|q_f(0)| = \frac{1}{2}|w(0)|$ on $B_r = \{(t,u,x)\,|\,\|(t,u,x)\|<r\}$. (To spare notation, we use the same symbol denoting balls in different spaces with the same radius.) This estimation can be done in the same manner as above by first estimating r by the first order derivatives of q_f using Taylor's formula, and then estimating the derivatives of q_f by those of f using A.3. From this we will find that there exist a positive polynomial $\bar{\bar{P}}>1$, an integer $\bar{\bar{m}}$ and a compact $\bar{\bar{M}}$ all independent of f such $|q_f(t,u,x)|>\frac{1}{2}|q_f(0)| = \frac{1}{2}|w(0)|$ on B_r, where $r = |w(0)|/\bar{\bar{C}}_f$ and $\bar{\bar{C}}_f = \bar{\bar{P}}(\|f\|_{\bar{\bar{m}},\bar{\bar{M}}})$. Taking $P = 2(\bar{P}+\bar{\bar{P}})$, $M = \bar{M}\cup\bar{\bar{M}}$, $m = \max(\bar{m},\bar{\bar{m}})$ and $C_f = P(\|f\|_{m,M})$, we see that $\bar{h}_g$, $\bar{q}_g$ are well defined on the neighbourhoods B_r^1, B_r^2 for $r = |w(0)|^p/C_f$.

Next, we need to estimate the derivatives of $\bar{h}_g$, $\bar{q}_g$. Recall that $\bar{h}_g^i(x) = h_g^i(\theta(0,x),x)$ and that $\bar{q}_g(t,x) = q_g(t,\theta(0,x),x)/q_f(t,\theta(0,x),x)$. The map θ was defined by $\phi^{-1}(u,x) = (\theta(u,x),x)$, where $\phi(u,x) = (h_f(u,x),x)$. Let us first sketch how to estimate the derivatives of θ. Recall that $|\text{Det } D\phi(0)| = |w(0)|^p$. So in a neighbourhood of 0, we have that $|\text{Det } D\phi(u,x)|>\frac{1}{2}|w(0)|^p$. The radius of a ball where this inequality is valid can be estimated by the second order derivatives of ϕ using Taylor's formula. We can use A.3 to estimate the derivatives of h_f, and this gives us estimates of the derivatives of ϕ as well. From these estimates, we find that $|\text{Det } D\phi(u,x)|>\frac{1}{2}|w(0)|^p$ on a ball with a radius equal to $|w(0)|^p/P(\|f\|_{m,M})$ for some new polynomial P, some new integer m and some new compact M. By adjusting our old P,M,m, we may suppose that $|\text{Det } D\phi(u,x)|>\frac{1}{2}|w(0)|^p$ on our neighbourhood B_r. The derivatives of ϕ^{-1} are fractions with polynomial expressions of the derivatives of ϕ in the nominator, and powers of $\text{Det } D\phi(u,x)$ in the denonominator. Using A.3 to estimate the derivatives of ϕ, and using that $|\text{Det } D\phi(u,x)|>\frac{1}{2}|w(0)|^p$ on B_r, we can find upper bounds for the derivatives of ϕ^{-1} in terms of f and negative powers of $w(0)$. Using this, and the fact that $q_f>\frac{1}{2}|w(0)|$ on B_r, and using A.3 to get upper bounds for the derivatives of q_g, h_g in terms of g, we can at last use the chain rule to estimate the derivatives of $\bar{h}_g$, $\bar{q}_g$. Writing this out in detail, we will obtain estimates which are of the type (2) and (3) of this propositon.

In the case $|w(0)|>1$, it is easy to see that we can ingnore the dependence of $|w(0)|$ in every estimate above, hence we can put all the exponents equal 0.

Proposition A.5. Let the notation be as in A.4. Then we can find a $\beta'>0$, and non negative integers m', m'', and a positive polynomial P', such that the following hold:

For every smooth function s on $R\times R^n$ such that

$$(1) \qquad \|s\|_{m',M} \leq \frac{|w(0)|^{\beta'}}{P'(\|f\|_{m'',M})},$$

there exist functions $\bar{\bar{h}}^i_g$, $i = 1,\ldots,p$, and a function $\bar{\bar{q}}^i_g$, defined on

$$B^1_{r/2} = \{x\in R^n \mid \|x\|<r/2\}, \quad B^2_{r/2} = \{(t,x)\in R\times R^n \mid \|(t,x)\|<r/2\}$$

respectively such that

$$(2) \qquad g(t,x) = \bar{\bar{q}}_g(t,x)(f(t,x)+s(t,x)) + \sum_{i=1}^{p} \bar{\bar{h}}^i_g(x)t^{p-1}$$

holds on $B^1_{r/2}$, $B^2_{r/2}$. (Here r is defined in A.4.)
Also

$$(3) \qquad \|\bar{\bar{h}}_g\|_{k,B^1_{r/2}} \leq C_{k,f+s}C_{k,g}|1/w(0)|^{\beta_k},$$

$$(4) \qquad \|\bar{\bar{q}}_g\|_{k,B^2_{r/2}} \leq C_{k,f+s}C_{k,g}|1/w(0)|^{\beta_k}$$

hold with $C_{k,f+s} = P_k(\|f+s\|_{m_k,M})$, $C_{k,g} = P_k(\|g\|_{m_k,M})$. In the case when $|w(0)|>1$, we must take $\beta' = 0$, otherwize the β', P' are independent of f only on n,p.

Remark A.6. A.5 corresponds to Proposition 2 of [2], p. 275. However the estimates (3), (4) will be needed for our purpose.

Proof of A.5. As in the proof of A.4 use A.1 to write

$$(f+s)(t,x) = \Gamma_p(t,u)q_{f+s}(t,u,x) + \sum_{i=1}^{p} h^i_{f+s}(u,x)t^{p-i},$$

$$g(t,x) = \Gamma_p(t,u)q_g(t,u,x) + \sum_{i=1}^{p} h_g^i(u,x)t^{p-i}.$$

Consider $\phi_s(u,x) = (h_{f+s}(u,x),x)$. Using a Taylor expansion of $(f+s)(t,0)$, we find

$$\det(D\phi_s(0)) = (-1)^p(q_{f+s}(0))^p = (-1)^p(w(0) + \frac{1}{p!}\frac{\partial^p}{\partial t^p} s(0))^p.$$

Since $w(0)\neq 0$, we have $|\det(D\phi_s(0))|>|\frac{1}{2}w(0)|^p \neq 0$, provided $|(1/p!)\frac{\partial^p s}{\partial t^p}(0)|<\frac{1}{2}|w(0)|$. Supposing this, ϕ_s is invertible in a neighbourhood of $\phi_s(0)$. Let $\phi_s^{-1}(u,x) = (\theta_s(u,x),x)$. Let us suppose that 0 belongs to the theighbourhood of $\phi_s(0)$ where ϕ_s^{-1} is defined. Put $\bar{\bar{h}}^i(x) = h_g^i(\theta_s(0,x),x)$. Now suppose $q_{f+s}(0,\phi_s^{-1}(0))\neq 0$, then in a neighbourhood of 0, define $\bar{\bar{q}}_g(t,x) = \dfrac{q_g(t,\theta_s(0,x),x)}{q_{f+s}(t,\theta_s(0,x),x)}$. Now since $h_{f+s}^i(\theta_s(0,x))\equiv 0$, we get that

$$g = (f+s)\bar{\bar{q}}_g + \sum_{i=1}^{p} \bar{\bar{h}}_g t^{p-i}$$

in a neighbourhood of 0.

We will now sketch the argument for the existense og $\bar{\bar{q}}_g$, $\bar{\bar{h}}_g$ provided β', P' is chosen suficiently large.

First suppose that $|w(0)|\leq 1$. Then if s satisfies (1) for sufficiently large β', P' and m', we get that $\det(D\phi_s(0))\neq 0$. (In fact we can take $P'>2/p!$, $\beta'\geq 1$, $m'\geq p$.)

Now consider $\lambda_s(u,x) = (u,x)-(D\phi_s(0))^{-1}\circ\phi$. We have that $\det(D\lambda_s(0)) = 0$, so in a ball at 0 with some radius r', we have that $|\det(D\lambda_s(0))|<\frac{1}{2}$. Again it follows from the proof of the inverse function theorem that ϕ_s^{-1} is defined on a ball at $\phi_s(0)$ with radius $r'/2$. The r' above can be estimated in terms of the second order derivatives of λ_s. In Proposition A.4, we did a corresponding estimation in the case $s = 0$, which gave us that $\phi^{-1} = \phi_0^{-1}$ was defined in a neighbourhood of $0 = \phi_0(0)$ of radius r where r was estimated in A.4. Now if P' is sufficiently large, then we can easily show that bounds for the derivatives of h_{f+s} can be approximated by bounds for the derivatives of h_f. (Using A.3 and the fact that h_f is linear in f.) By choosing m', β', P' sufficiently large, we also get that $\det(D\phi_s(0))$ is close to $\det D\phi_0(0)$. This will imply that bounds for the second order derivatives of λ_s can be approximated as good as we want (by choosing m', P', β' large enough) with bounds for the second order derivatives of λ_0. This will imply that we can get $r'/2$ as close to the r of A.4 as we want by choosing P', β' large enough. Also by the same reasoning, we can get $\phi_s(0)$ as close to $\phi_0(0) = 0$ as we want. We may therefore obtain that the ball at 0 with radius $r/2$ is contained in the ball at $\phi_s(0)$ with radius $r'/2$ (since $r'/2$ is close to r, $\phi_s(0)$ is close to 0) provided m', P', β' are chosen large enough.

Recall that our definition of $\bar{\bar{q}}_g$ requires that $q_{f+s} \neq 0$. Also, recall that we get from the proof of the inverse function theorem in [4], that ϕ_s^{-1} restricted to the ball at $\phi_s(0)$ with radius $r'/2$ takes values in the ball at 0 with radius r'. In the proof of A.4 we had that $|q_f(0)| = |w(0)|$, and that $|q_f| > \frac{1}{2}|w(0)|$ on $B_{2r} = \{(x,u) \mid \|(x,u)\| < 2r\}$. Now, we have that $r'/2$ can be approximated by r, and that $q_{f+s} = q_f + q_s$ (which follows from the linearity of Q in A.1). Using A.3, we can approximate q_s and obtain that $|q_{f+s}(t,\theta(0,x),x)| > \frac{1}{4}|w(0)|$ on the smaller ball $B_{r/2}$. (All this can be achieved by choosing m', P', β' large enough.) This will show that $\bar{\bar{q}}_g$, $\bar{\bar{h}}_g^i$ are well defined on $B_{r/2}$.

To prove the estimates (3) and (4), we just reason as in A.4 and use that $|q_{f+s}(t,\theta(0,x),x)|>\frac{1}{4}|w(0)|$ on $B_{r/2}$, and that we also can prove (by a similar argument) that $|\det(D\phi_s((\theta(0,x),x))|>\frac{1}{4}|w(0)|^p$ on $B_{r/2}$ provided m', β', P' are large enough. Now can estimate the derivatives of $\bar{\bar{q}}_g$, $\bar{\bar{h}}^i_g$ using Taylor's formula, the chain rule, and A.3, as we did in A.4, and also using that the derivatives of $f+s$ are close to those of f provided m', β', P' are large enough. We will end up with estimates of the type (3) and (4). To get the same polynomial P_k to hold in both A.4 and A.5, it is perhaps necessary to choose the P_k in A.4 some what larger.

In the case $|w(0)|>1$, we can reason as above but ignore the dependence of $w(0)$, hence take all exponent to be 0.

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